Week Five

The Econometrics of Asset Pricing Relations

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The Semiparametric Nature of Asset Pricing Models

- Two period stochastic discount factor models
  - $E_P[m_{t+1}R_{it+1}|\mathscr{F}_t] = 1$
  - $\Rightarrow m_{t+1}R_{it+1} = 1 + g_{it+1}$; $E[g_{it+1}|\mathscr{F}_t] = 0$
- Asset pricing models
  - $m_t = m_t[\{x_{t-s}, s\geq 0\}, \theta]$; $x_t \in \mathscr{F}_t$; $\theta = \{\theta_1, \ldots, \theta_p\}'$
- $\Rightarrow$ GMM is natural analogue estimation method if rational expectations implicit in P
  - *Ex ante* = *ex post* moments in large samples
  - Need observable beliefs or explicit model for nonrational beliefs

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Some Unconditional Moments of Asset Pricing Relations

- Martingale difference errors
  - $0 = E[g_{it+1}|\mathscr{F}_t] = E[g_{it+1}] = E[m_{t+1}R_{it+1}-1]$
- Sample mean moment conditions
  - $g_T(\theta) = \sum_t g_t(\theta)/T;\ g_t(\theta) = m_t R_t - \iota$
    - $m_t = m_t(x_t, \theta);\ R_t=\{R_{1t},\ldots,R_{Nt}\}'$
- Population mean moment conditions
  - $g_T(\theta_0) = \sum_t g_t(\theta_0)/T;\ E[g_T(\theta_0)] = \sum_t E[g_t(\theta_0)]/T = 0$
  - $E[g_T(\theta_0)g_T(\theta_0)'] = E[\sum_t g_t(\theta_0)\sum_t g_t(\theta_0)'/T]/T$
    $= E[\sum_t \sum_s g_t(\theta_0)g_s(\theta_0)'/T]/T$
    $= E[\sum_t g_t(\theta_0)g_t(\theta_0)'/T]/T \equiv S_T(\theta_0)/T$

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Unconditional GMM Estimation of Asset Pricing Relations

- If N>p, $\nexists\ \theta_T$ s. t. $g_T(\theta_T) = 0$ and so:
- Minimization of quadratic form in $g_T(\theta)$
  - $\min_\theta g_T(\theta)'W_T(\theta_{T*})g_T(\theta)$; $|W_T(\theta_{T*})| > 0$
    - $\theta_{T*} \rightarrow \theta_0 \Rightarrow W_T(\theta_{T*}) \rightarrow W(\theta_0)$; $|W(\theta_0)| > 0$
  - Solution: $G_T(\theta_T)W_T(\theta_{T*})g_T(\theta_T) = 0$
    - $G_T(\theta) = \partial g_T(\theta)'/\partial\theta$
- Alternative: Choose pxN matrix $A_T(\theta_{T*})$ s. t. $A_T(\theta_{T*}) \rightarrow A(\theta_0)$ and solve $A_T(\theta_{T*})g_T(\theta_T) = 0$
  - Same as above when $A_T(q_{T*}) = G_T(q_T)W_T(q_{T*})$
  - Portfolio interpretation: $R_{At}(q_{T*}) = A_T(q_{T*})R_t$

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Asymptotic Properties of GMM Estimators

- Identification
  - $G_T(\theta_0)$ has full row rank
  - $S_T(\theta_0) \rightarrow S(\theta_0)$; $|S(\theta_0)| > 0$
- Weak law of large numbers insures
  - $g_T(\theta_0) \rightarrow 0 \Rightarrow \theta_T \rightarrow \theta_0$
  - $G_T(\theta_0) \rightarrow G(\theta_0)$
- First order expansion of $g_T(\boldsymbol{q}_T)$
  - $g_T(\theta_T) = g_T(\theta_0) + G_T(\theta_0)'(\theta_T - \theta_0) + O_p \| \theta_T - \theta_0 \|$
    - $O_p \| \theta_T - \theta_0 \| = O_p(1/T)$

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Asymptotic Properties of GMM Estimators, Continued

- First order expansion and Slutsky's theorem
  - $\sqrt{T}\,(\theta_T - \theta_0) \rightarrow -\sqrt{T}\,D(\theta_0)\,g_T(\theta_0)$
    - $D(\theta_0) = [G(\theta_0)S(\theta_0)^{-1}G(\theta_0)']^{-1}G(\theta_0)W(\theta_0)$
- Associated convergence in distribution
  - $\sqrt{T}\,(\theta_T - \theta_0) \rightarrow N[0, D(\theta_0)S(\theta_0)D(\theta_0)']$
- Efficient choice of weighting matrix $W_T(\theta_0)$
  - $W_T(\theta_0) = S_T(\theta_0)^{-1} \rightarrow S(\theta_0)^{-1}$
  - $\Rightarrow \sqrt{T}\,(\theta_T - \theta_0) \rightarrow N[0, (G(\theta_0)S(\theta_0)^{-1}G(\theta_0)')^{-1}]$
  - Asymptotic efficiency defined in terms of asymptotic variance

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Hansen's J Test

- Large sample behavior of $g_T(\theta_0)$
  - $\sqrt{T} g_T(\theta_0) \rightarrow N[0, S(\theta_0)]$
  - $\Rightarrow T g_T(\theta_0)' S(\theta_0)^{-1} g_T(\theta_0) \rightarrow \chi^2(N)$
- Large sample behavior of $g_T(\theta_0)$
  - $\sqrt{T} g_T(\theta_0) \rightarrow N[0, S(\theta_0)]$
  - $\sqrt{T} g_T(\theta_T) \rightarrow N[0, V(\theta_0)]$
  - $V(\theta_0) = S(\theta_0) - G(\theta_0)'(G(\theta_0) S(\theta_0)^{-1} G(\theta_0)')^{-1} G(\theta_0)$
    - Correction for degrees of freedom used in estimation
  - $\Rightarrow T g_T(\theta_T)' S(\theta_{T*})^{-1} g_T(\theta_T) \rightarrow \chi^2(N\text{-}p)$
  - Possible to construct other tests

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Maximum Likelihood Interpretation of Efficient GMM

- Let:
  - $\mathcal{L}_t(\theta_0,\eta_0)$=ln p($y_t|\theta_0,\eta_0$)=ln p($y_t$|•) be stationary or unconditional log likelihood of data y_t in $g_t(\theta_0)$
    - *h* is possibly infinite-dimensional nuisance parameter
  - $\ell_t(\theta_0,\eta_0) = \partial\mathcal{L}_t(\theta_0, \eta_0)/\partial\theta$ be score function for θ evaluated at θ_0
    - Assume well-behaved score in neighborhood of θ_0 to permit interchange of differentiation and integration
- Projection of score on moment conditions
  - $\ell_t(\theta_0,\eta_0) = E[\ell_t(\theta_0,\eta_0)g_t(\theta_0)']S(\theta_0)^{-1}g_t(\theta_0) + \omega_{ut}$

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Maximum Likelihood Interpretation, Continued

- The population value of $E[\ell_t(\theta_0,\eta_0)g_t(\theta_0)']$
  - $E[\ell_t(\theta_0,\eta_0)g_t(\theta_0)'] = \int \ell_t(\theta_0,\eta_0)g_t(\theta_0)'\, p(y_t|\bullet)\, dy_t$
  - $p(y_t|\bullet)\ell_t(\theta_0,\eta_0) = \partial p(y_t|\bullet)/\partial\theta \; [p(y_t|\bullet)/p(y_t|\bullet)]$
    $= \partial p(y_t|\theta_0,\eta_0)/\partial\theta$
  - $\Rightarrow E[\ell_t(\theta_0,\eta_0)g_t(\theta_0)'] = \int [\partial p(y_t|\bullet)/\partial\theta]\; g_t(\theta_0)'\, dy_t$
    $= [\partial/\partial\theta] \int g_t(\theta_0)'\, p(y_t|\bullet)\, dy_t$
    $= \int [\partial g_t(\theta_0)'/\partial\theta]\, p(y_t|\bullet)\, dy_t$
    $= E[G_T(\theta_0)]$
- Projection of score on moment conditions
  - $\Rightarrow \ell_t(\theta_0,\eta_0) = E[G_T(\theta_0)]S(\theta_0)^{-1}g_t(\theta_0) + \omega_{ut}$
    $= A(\theta_0)g_t(\theta_0) + \omega_{ut}$

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Maximum Likelihood Interpretation, Continued

- Efficient GMM uses linear combination of moment conditions that has largest unconditional correlation with the true, but unknown, conditional score in *finite* samples
  - No obvious finite sample efficiency properties for either the MLE or efficient GMM
- Portfolios with weights $A(\theta_0)$ and cost $A(\theta_0)\iota$ has payoffs with largest unconditional correlations with population scores $\ell_t(\theta_0,\eta_0)$
  - That is, they are the optimal hedge portfolios

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(Generally Infeasible) Conditional GMM

- Arithmetic similar to unconditional case
  - Identification: $G_t(\theta_0) = \partial g_t(\theta_0)'/\partial\theta$ has full row rank and $|S_t(\theta_0)| > 0$
  - Choose pxN matrix $A_t(\theta_{T*})$ s. t. $A_t(\theta_{T*}) \rightarrow A_t(\theta_0)$
  - Solve $\sum_t A_t(\theta_{T*})g_t(\theta_T)=\sum_t G_t(\theta_T)W_t(\theta_{T*})g_t(\theta_T)=0$
  - Efficient $W_t(\bullet) = E[\sum_t g_t(\bullet)g_t(\bullet)'|\mathscr{F}_{t-1}] = S_t(\bullet)$
- First order asymptotics
  - $\sqrt{T}\,(\theta_T-\theta_0) \rightarrow -\sqrt{T}\sum_t D_t(\theta_0)\,g_t(\theta_0)$
    - $D_t(\theta_0) = [G_t(\theta_0)S_t(\theta_0)^{-1}G_t(\theta_0)']^{-1}\,G_t(\theta_0)S_t(\theta_0)^{-1}$
  - $\sqrt{T}\,(\theta_T-\theta_0) \rightarrow N[0, \sum_t (G_t(\theta_0)S_t(\theta_0)^{-1}G_t(\theta_0)')^{-1}]$

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Efficient Conditional GMM and Maximum Likelihood

- Let $\mathcal{L}_t(\theta_0,\eta_0) = \ln p(y_t|\mathscr{F}_{t-1},\theta_0,\eta_0)$ be the conditional log likelihood of data y_t in $g_t(\theta_0)$ with score $\ell_t(\theta_0,\eta_0) = \partial\mathcal{L}_t(\theta_0,\eta_0)/\partial\theta$ as before
- Corresponding conditional projection is:
  - $\Rightarrow \quad \ell_t(\theta_0,\eta_0) = E[G_t(\theta_0)|\mathscr{F}_{t-1}]S_t(\theta_0)^{-1}g_t(\theta_0)+\omega_{ct}$
    $= A_t(\theta_0)g_t(\theta_0) + \omega_{ct}$
- Optimal hedge portfolios for conditional score have weights $A_t(\theta_0)$ and cost $A_t(\theta_0)\,\iota$
  - i.e., payoffs with largest conditional correlations with population scores $\ell_t(\theta_0,\eta_0)$

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Feasible Conditional GMM Via Trading Strategies

- Fix information $Z_{t-1} \in \mathscr{F}_{t-1}$
- Population moment conditions
  - $Z_{t-1}\, E[R_t m_t | \mathscr{F}_{t-1}] = E[Z_{t-1} R_t m_t | \mathscr{F}_{t-1}] = Z_{t-1}\iota$
  - $E[Z_{t-1} R_t m_t] = E(Z_{t-1}\iota)$
    - Average risk-adjusted return equals average cost
  - Termed trading strategies because returns are given different weights temporally and cross -sectionally after the fashion of active investors
- $\Rightarrow$ Apply GMM to $g_t(\theta) = Z_{t-1}(R_t m_t - \iota)$
- N. B.: No pretense of conditional efficiency

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Smooth Approximation of Stochastic Discount Factors

- Recall SDF models take the form:
  - $m_t = m_t[\{x_{t-s}, s \geq 0\}, \theta]; \; x_t \in \mathscr{F}_t; \; \theta = \{\theta_1, \ldots, \theta_p\}'$
- Suppose $m_t = m(x_t, x_{t-1}, t, \theta)$ is smooth and that x_t varies smoothly as with diffusions
  - $$\begin{aligned} m(x_t, x_{t-1}, t, \theta) = {} & 1 + m_3(x_{t-1}, x_{t-1}, t-1, \theta) \\ & + \Delta x_t' m_{11}(x_{t-1}, x_{t-1}, t-1, \theta) \Delta x_t / 2 \\ & + m_1(x_{t-1}, x_{t-1}, t-1, \theta)' \Delta x_t + o[\Delta t, \Delta x_t] \\ \approx {} & \kappa_{t-1} + \Phi_{t-1}' \Delta \ln x_t \end{aligned}$$
    - $\kappa_{t-1} = 1 + m_3(x_{t-1}, x_{t-1}, t-1, \theta) + V_{11t-1}; \; m(x_{t-1}, x_{t-1}, t-1, \theta) = 1;$
      $V_{11t-1} = E_{t-1}[\Delta x_t' m_{11}(x_{t-1}, x_{t-1}, t-1, \theta) \Delta x_t / 2]$
      $\Phi_{t-1} = \mathrm{Diag}[x_{t-1}] \, m_1(x_{t-1}, x_{t-1}, t-1, \theta)$

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State-Variable-Driven Stochastic Discount Factor Models

- Revised riskless rate valuation
  - $1 = E\{R_{ft}[\kappa_{t-1} + \Phi_{t-1}'\Delta\ln x_t]|\mathscr{F}_{t-1}\}$
    $\approx R_{ft}[\kappa_{t-1} + \Phi_{t-1}'\Delta\mu_{xt-1}]$
    - $\Delta\mu_{xt-1} \equiv E[\Delta\ln x_t|\mathscr{F}_{t-1}]$
  - $\Rightarrow R_{ft} \approx [\kappa_{t-1} + \Phi_{t-1}'\Delta\mu_{xt-1}]^{-1}$
- Revised risky asset valuation
  - $0 = E[(R_{it}-R_{ft})m_t]|\mathscr{F}_{t-1}]$
    $= R_{ft}^{-1}E[R_{it}-R_{ft}|\mathscr{F}_{t-1}] + Cov[R_{it},m_t|\mathscr{F}_{t-1}]$
  - $\Rightarrow E[R_{it}-R_{ft}|\mathscr{F}_{t-1}] \approx -R_{ft}\Phi_{t-1}'Cov[R_{it}, \Delta\ln x_t|\mathscr{F}_{t-1}]$

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Conditional Linear Factor Pricing Models

- Project excess returns on $\Delta \ln x_t$
  - $R_{it} - R_{ft} = \alpha_{ixt} + \beta_{ixt-1}' \Delta \ln x_t + \varepsilon_{ixt}$
  - $E(\varepsilon_{ixt} | \mathscr{F}_{t-1}) = E(\Delta \ln x_t \, \varepsilon_{ixt} | \mathscr{F}_{t-1}) = 0$
- Revised risky asset valuation
  - $R_{it} - R_{ft} = \beta_{ixt-1}' (\Delta \ln x_t - \lambda_{xt-1}) + \varepsilon_{ixt}$
  - $\beta_{ixt-1} = \mathrm{Var}[\Delta \ln x_t | \mathscr{F}_{t-1}]^{-1} \, \mathrm{Cov}[R_{it}, \Delta \ln x_t | \mathscr{F}_{t-1}]$
  - $\lambda_{xt-1} = \Delta \mu_{xt-1} + R_{ft} \, \mathrm{Var}[\Delta \ln x_t | \mathscr{F}_{t-1}]^{-1} \, \Phi_{t-1}$
- N. B.: ε_{ixt} processes need not be smooth
  - i.e., ε_{ixt} can have (unpriced) discontinuous jumps

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Risk Premiums in Conditional Linear Factor Pricing Models

- Consider cross-sectional regression of excess returns on β_{ixt-1} via WLS
  - $R_{it} – R_{ft} = \beta_{ixt-1}'(R_{xt} – R_{ft}) + \eta_{ixt}$; $E(\eta_{ixt}|\mathscr{F}_{t-1}) = 0$
  - $R_{xt} – \iota R_{ft} = P_{B_{xt-1}}(R_t – \iota R_{ft}) = \Delta \ln x_t – \lambda_{xt-1} + P_{B_{xt-1}} \varepsilon_{xt}$
  - $P_{B_{xt-1}} = [B_{xt-1}'A_{xt-1}^{-1}B_{xt-1}]^{-1}B_{xt-1}'A_{xt-1}^{-1}$
    - $\lim_{N\to\infty}\xi_{min}(B_{xt-1}'A_{xt-1}^{-1}B_{xt-1}) = \infty \Rightarrow \lim_{N\to\infty}P_{B_{xt-1}}\varepsilon_{xt} = 0$
  - $A_{xt-1}=\Omega_{xt-1}=Var(\varepsilon_{xt}|\mathscr{F}_{t-1})$ yields the GLS estimate
- Risk premiums and CSR coefficients
  - $E(R_{xt} – \iota R_{ft}|\mathscr{F}_{t-1}) = \Delta \ln x_t – \lambda_{xt-1} + P_{B_{xt-1}} E(\varepsilon_{xt}|\mathscr{F}_{t-1})$
    $= –R_{ft}\, Var[\Delta \ln x_t|\mathscr{F}_{t-1}]^{-1}\, \Phi_{t-1}$

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Strengths and Weaknesses of Linear Factor Pricing Model

- Strengths
  - Relies only on perfect markets and identification of pricing relevant states with state variables
  - No explicit assumptions on preferences or other traded and nontraded assets
- Weakness
  - Φ_{t-1} unmodeled $\Rightarrow$ risk premium λ_{xt-1} unrestricted
    - Empirically costly given imprecision of sample means
  - Only relative asset prices determined by absence of arbitrage alone

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The Interpretation of Factor Risk Premiums

- Arguments about risk premiums have no meaning in no-arbitrage framework
  - No restrictions on sign or magnitude
  - Nonzero risk premium does not mean market rationally prices some economic risk
- Question not if market reacts to relevant risks but if response is economically appropriate
  - Answer requires model for risk premiums
  - i.e., plausible arguments about how investors *feel* about the associated risks

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The Information Requirements of Linear Factor Pricing Models

- Inputs
  - Conditional moments
    - B_{xt-1}, Ω_{xt-1}, and λ_{xt-1}
  - Sample $\{R_t, R_{ft}, \Delta x_t; 1,\ldots,T\}$
- Convenient parameterizations in absence of theory of conditional moments
  - Constant parameters
  - Linearity in conditioning information
  - Flexible models with large parameter spaces
    - Often based on financial ratios used by analysts

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Trading Strategies and Time -Varying Risk Exposures

- In matrix form:
  - $R_t - \iota R_{ft} = B_{xt-1}(\Delta \ln x_t - \lambda_{xt-1}) + \varepsilon_{xt}$; $E(\varepsilon_{xt} | \mathscr{F}_{t-1}) = 0$
- Project B_{xt-1} on $Z_{t-1} \in \mathscr{F}_{t-1}$ (with an intercept)
  - $B_{xt-1} = Z_{t-1}\Lambda_{\beta_{t-1}} + \zeta_{xt-1}$; $Z_{t-1}' A_{zt-1} \zeta_{xt-1} = 0$
- Form the mimicking or basis portfolios with weights $P_{Z_{t-1}} = (Z_{t-1}' A_{zt-1} Z_{t-1})^{-1} Z_{t-1}' A_{zt-1}$; $|A_{zt-1}| > 0$
  - $R_{zt} - \iota R_{ft} = (Z_{t-1}' A_{t-1} Z_{t-1})^{-1} Z_{t-1}' A_{t-1}(R_t - \iota R_{ft})$
    $= P_{Z_{t-1}}[(Z_{t-1}\Lambda_{\beta_{t-1}} + \zeta_{xt-1})(\Delta \ln x_t - \lambda_{xt-1}) + \varepsilon_{xt}]$
    $= \Lambda_{\beta_{t-1}}(\Delta \ln x_t - \lambda_{xt-1}) + P_{Z_{t-1}}\varepsilon_{xt}$
  - $E(P_{Z_{t-1}}\varepsilon_{xt} | \mathscr{F}_{t-1}) = P_{Z_{t-1}} E(\varepsilon_{xt} | \mathscr{F}_{t-1}) = 0$

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Risk Premiums for Time-Varying Risk Exposures

- Basis portfolios and risk/return relations
  - $R_t - \iota R_{ft} = B_{xt-1}(\Delta \ln x_t - \lambda_{xt-1}) + \varepsilon_{xt}$
    $= Z_{t-1}(R_{xt} - \iota R_{ft}) + \psi_{xt}$
  - $\Rightarrow E(R_t - \iota R_{ft} | \mathscr{F}_{t-1}) = Z_t E(R_t - \iota R_{ft} | \mathscr{F}_{t-1}) + E(\psi_{xt} | \mathscr{F}_{t-1})$
  - $\Rightarrow E(\psi_{xt} | \mathscr{F}_{t-1}) = \zeta_{xt-1}[\Delta \mu_{xt-1} - \lambda_{xt-1}]$
- Conditional and unconditional pricing errors approximately zero if WLLN applies to ζ_{xt-1}
  - Diversifiability arises when $\zeta_{xt-1}' \zeta_{xt-1} \rightarrow k < \infty$
  - i.e., APT-like approximate linearity due to approximate linear model for time-varying betas

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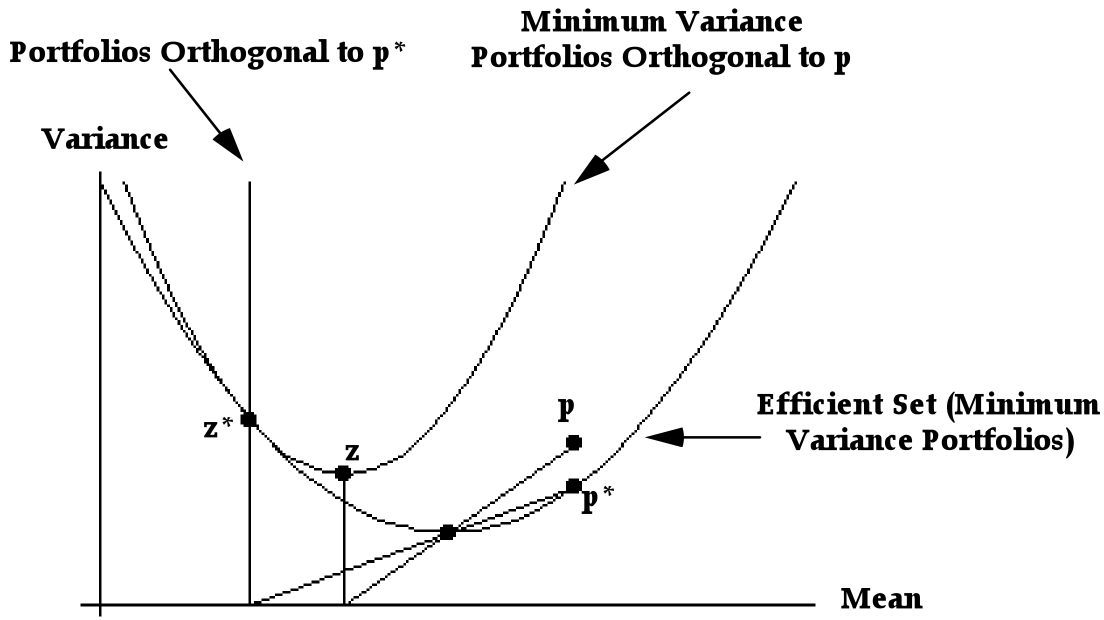
The Intuition Behind Mean -Variance Efficiency Tests



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Mean-Variance Efficiency Test Intuition, Continued

- All zero beta portfolios relative to MV efficient p* have identical expected returns
  - All costless zero beta portfolios relative to p* have zero expected returns
- ∃ zero beta portfolios relative to inefficient p at all levels of expected return
  - Costless zero beta portfolios need not have zero expected returns when p is inefficient
- MV efficiency tests ask if specific costless zero beta portfolios have zero mean returns

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The Simplest Case

- One security in the CAPM
  - $R_{it} = \alpha_{im} + (1-\beta_{im})R_{ft} + \beta_{im}R_{mt} + \varepsilon_{it}$
  - $E[R_i] = \alpha_{im} + (1-\beta_{im})R_{ft} + \beta_{im}E[R_{mt}]$
- Economic interpretation of $t(\alpha_{im})$
  - Buy \$1 of security i
  - Sell short $\$(1-\beta_{im})$ of the riskless asset
  - Sell short $\$\beta_{im}$ of the market portfolio
  - Portfolio is costless and has a zero beta
- $\Rightarrow$ t-statistic for $\alpha_{im} = 0$ tests for joint mean -variance efficiency of m and f

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Preliminaries

- Notation:
  - R = vector of returns on N given assets
  - $\mathcal{R}$ = associated vector of expected returns
  - V = associated return covariance matrix
  - β_p = regression coefficients of R on p
  - $\mathcal{R}_p$ = expected return of p
  - σ_p^2 = variance of return of p
  - ι = suitably conformable vector of ones
- All moments may be conditional but sample analogues typically for iid returns

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A Virtuous but Usually Infeasible Costless Zero Beta Portfolio

- The maximum squared Sharpe ratio costless zero beta portfolio
  - $\min_{w_x} w_x'Vw_x$ s.t. $w_x'\iota = 0$; $w_x'\mathcal{R} = \mathcal{R}_x$; $w_x'\beta_p = 0$
- Solution
  - Weights: $w_x \propto V^{-1}\alpha_p$; $\alpha_p'V^{-1}\iota = \alpha_p'V^{-1}\beta_p = 0$
  - Squared Sharpe ratio: $\alpha_p'V^{-1}\alpha_p$
  - $\mathcal{R} = \alpha_p + \iota\,\mathcal{R}_z + \beta_p\,(\mathcal{R}_p - \mathcal{R}_z)$
    - $\mathcal{R}_z$ solves $\min_{w_z} w_z'Vw_z$ s.t. $w_z'\iota = 0$; $w_z'\beta_p = 0$
- Sample analogue infeasible when N>T

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Portfolio Grouping and Feasible Costless Zero Beta Portfolios

- Group assets into M portfolios via W_M
  - i.e., $\mathcal{R}_M = W_M'\mathcal{R}$; $V_M = W_M'VW_M$; $\beta_{pM}=W_M'\beta_p$
- Grouping and the maximum squared Sharpe ratio costless zero beta portfolio
  - Weights: $w_{xM} \propto V_M^{-1}\alpha_{pM}$; $\alpha_{pM}'V_M^{-1}\iota = 0$
    $\alpha_{pM}'V_M^{-1}\beta_{pM} = 0$
  - $\mathcal{R}_M = \alpha_{pM} + \iota\,\mathcal{R}_{zM} + \beta_{pM}(\mathcal{R}_p - \mathcal{R}_{zM})$
    - $\mathcal{R}_{zM}$ solves $\min_{w_{zM}} w_{zM}'V_M w_{zM}$ s.t. $w_{zM}'\iota = 0$;
      $w_{zM}'\beta_{pM} = 0$
  - Squared Sharpe ratio: $\alpha_{pM}'V_M^{-1}\alpha_{pM}$

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Why Portfolio Grouping Destroys the Optimality of Test Statistics

- Relation between $\alpha_p' V^{-1} \alpha_p$ and $\alpha_{pM}' V_M^{-1} \alpha_{pM}$
  - $\alpha_{pM}' V_M^{-1} \alpha_{pM} = \alpha_{pM*}' V_M^{-1} \alpha_{pM*} - (\iota - \beta_{pM})' V_M^{-1} (\iota - \beta_{pM}) (\mathcal{R}_z - \mathcal{R}_{zM})^2$
  - where α_{pM*} = vector of SML deviations of grouped portfolios w.r.t. to $\mathcal{R}_z$ instead of $\mathcal{R}_{zM}$
  - Cannot simplify further without additional information on W_M, α_{pM}, $(\mathcal{R}_z - \mathcal{R}_{zM})$, and V_M
  - Sample analogue similarly immune to analytical comparisons

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Minimum Variance Costless Zero Beta Portfolios

- The maximum squared Sharpe ratio costless zero beta portfolio
  - $\min_{w_x} w_x' V w_x$ s.t. $w_x' \iota = 0$; $w_x' \mathcal{R} = \mathcal{R}_x$; $w_x' \beta_p = 0$
  - Solution: $w_x \propto V^{-1} \alpha_p$; $\alpha_p' V^{-1} \iota = \alpha_p' V^{-1} \beta_p = 0$
    - Note proportionality makes $\mathcal{R}_x$ a scale factor
    - Squared Sharpe ratio is $\alpha_p' V^{-1} \alpha_p$
- Expected return relations
  - $\mathcal{R} = \alpha_p + \iota \mathcal{R}_z + \beta_p (\mathcal{R}_p - \mathcal{R}_z)$
    - $\mathcal{R}_z$ solves $\min_{w_z} w_z' V w_z$ s.t. $w_z' \iota = 0$; $w_z' \beta_p = 0$
  - $\mathcal{R} = \alpha_x \mathcal{R}_x + \iota \mathcal{R}_z + \beta_p (\mathcal{R}_p - \mathcal{R}_z)$; $\alpha_x = \alpha_p \sigma_p^2$

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Performance Improvement Via Costless Zero Beta Portfolios

- Portfolios p, z, and x:
  - are uncorrelated so variance of a portfolio of these portfolios is linear combination of the variances
  - $\Rightarrow$ span the mean-variance efficient set of these N+1 securities
    - $\forall\ \omega$, $w_\omega = \omega w_p + (1-\omega) w_z + \xi_\omega V^{-1}\alpha_p$ is mean-variance efficient
      - $\xi_\omega = [\omega\sigma_p^2 + (1-\omega)\,\sigma_z^2]/(\mathcal{R}_p - \mathcal{R}_z)$
    - $\mathcal{R}_\omega = \omega\mathcal{R}_p + (1-\omega)\mathcal{R}_z + \xi_\omega\alpha_p' V^{-1}\alpha_p$
    - $\sigma_\omega^2 = \omega^2\sigma_p^2 + (1-\omega)^2\sigma_z^2 + \xi_\omega^2\alpha_p' V^{-1}\alpha_p$

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Sampling Theory for the IID Case

- Sample multivariate excess return regression
  - $R_t - \iota R_{ft} = \alpha_m + b_m(R_{mt} - R_{ft}) + \varepsilon_t$; $S_m = \mathrm{Var}(\varepsilon_t)$
  - a, b, and S are sample analogues of α_m, β_m, and Σ
- Excess returns jointly NID implies
  - $[(T-N-1)/N]a_m' S_m^{-1} a_m / [1+(R_m - R_f)^2/s_m^2] \sim F_{N,T-N-1}$
- Hansen's J test can be used in the INID and stationary cases applied to linear or nonlinear asset pricing models

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