Week Three

Alternative Representations of State Prices
Diversification and Asset Prices

Valuation Based on Stochastic Discount Factors (SDFs)

- $m_s \equiv \psi_s/\pi_s$ = state price per unit probability = stochastic discount factor (SDF)
  - $\{\pi_s, s = 1,\ldots,S\}$ are arbitrary probabilities
  - Treat $\{\psi_s/\pi_s, s = 1,\ldots,S\}$ as realizations of a random variable
    - N. B.: State prices and SDFs need not be unique $\Rightarrow$ all that follows holds for *all marginal investor* SDFs
- Valuation relation
  - $p_i = \Sigma_s \pi_s v_{is} m_s = E_\pi[v_i m]$
  - $E_\pi[R_i m] = \Sigma_s \pi_s R_{is} m_s = \iota$
    - $\iota \equiv (1,\ldots,1)'$; $R_i \equiv v_i/p_i$; $R_{is} \equiv v_{is}/p_i$

SDFs and the Hansen-Jagannathan Bound

- Consider population projection of an SDF on unexpected returns from this asset menu
  - $m - E_\pi(m) = Cov_\pi(m,R)' Var(R)^{-1}[R - E_\pi(R)] + \varepsilon_m$
  - $Var_\pi(m) = Cov_\pi(m,R)' Var(R)^{-1} Cov_\pi(m,R) + Var(\varepsilon_m)$
- Implications for the projection variance
  - $Var_\pi(m) = E_\pi(m)^2 [E_\pi(R) - \iota R_f]' Var(R)^{-1} [E_\pi(R) - \iota R_f] + Var(\varepsilon_m)$
  - $Var_\pi(m) \geq E_\pi(m)^2 [E_\pi(R) - \iota R_f]' Var(R)^{-1} [E_\pi(R) - \iota R_f] \geq [E_\pi(R) - \iota R_f]' Var(R)^{-1} [E_\pi(R) - \iota R_f]$

Stochastic Discount Factors and the Efficient Frontier

- Recall projection of SDF on returns:
  - $m - E_\pi(m) = Cov_\pi(m,R)' Var(R)^{-1}[R - E_\pi(R)] + \varepsilon_m$
    $= -E_\pi(m)\,(\mathcal{R} - \iota\mathcal{R}_f)'\,V^{-1}[R - \mathcal{R}] + \varepsilon_m$
    $= -E_\pi(m)\,(\mathcal{R}_p - \mathcal{R}_f)\,\beta_p' V^{-1}[R - \mathcal{R}] + \varepsilon_m$
- $\Rightarrow$ Minimum variance hedge portfolio for m
  - $V^{-1}(\mathcal{R} - \iota\mathcal{R}_f) = V^{-1}\beta_p(\mathcal{R}_p - \mathcal{R}_f) = w_p(\mathcal{R}_p - \mathcal{R}_f)/\sigma_p^2$
    $= (\sigma_0^2/\sigma_p^2)\,[(\mathcal{R}_p - \mathcal{R}_f)/(\mathcal{R}_0 - \mathcal{R}_f)]V^{-1}(\mathcal{R} - \iota\mathcal{R}_f)$
    - $\Rightarrow (\mathcal{R}_p - \mathcal{R}_f)/\sigma_p^2 = (\mathcal{R}_0 - \mathcal{R}_f)/\sigma_0^2$
  - Costless maximum squared Sharpe ratio portfolio
    - Mean and variance $= [E_\pi(R) - \iota R_f]' Var(R)^{-1}[E_\pi(R) - \iota R_f]$

Stochastic Discount Factors and Beta Pricing Models

- Project returns on m
  - $R_i = \alpha_{im} + \beta_{im} m + \varepsilon_{im}$; $E_\pi[\varepsilon_{im}] = E_\pi[m\varepsilon_{im}] = 0$
- SDF covariance manipulations
  - $1 = E_\pi[R_i m] = \alpha_{im} E_\pi[m] + \beta_{im} E_\pi[m^2]$
  - $\Rightarrow R_f = \alpha_{im} + \beta_{im} R_f E_\pi[m^2] \Rightarrow \alpha_{im} = R_f[1 - \beta_{im} E_\pi(m^2)]$
- Implications for projection
  - $R_i - R_f = \beta_{im}[m - R_f E_\pi(m^2)] + \varepsilon_{im}$
    - $E_\pi(m^2) = E_\pi(m)^2 + Var_\pi(m)$
  - $\Rightarrow R_i - R_f = \beta_{im}[m - E_\pi(m) - R_f Var_\pi(m)] + \varepsilon_{im}$

Risk Premiums in the Beta Pricing Formulation

- Expected excess returns
  - $E_\pi[R_i - R_f] = \beta_{im} E_\pi\{[m - E_\pi(m)] - R_f Var_\pi(m)\}$
    $= -\beta_{im} R_f Var_\pi(m)$
- Related SDF covariance manipulations
  - $E_\pi[R_i\, m] = E_\pi[R_i]\, E_\pi[m] + Cov_\pi[R_i,m] = 1$
  - $\Rightarrow E_\pi[R_i] = [1 - Cov_\pi(R_i, m)]/E_\pi[m]$
    $= R_f [1 - Cov_\pi(R_i, m)]$
  - $\Rightarrow E_\pi[R_i - R_f] = -R_f Cov_\pi(R_i, m)$
    $= -\beta_{im} R_f Var_\pi(m)$
- $-R_f Var_\pi(m)$ = price of m risk

An Introduction to Risk Neutral Probabilities

- The (shadow) riskless rate
  - $p_f = R_f^{-1} = (1+r_f)^{-1} = \Sigma_s\ \psi_s$
- Risk neutral probabilities
  - $\{q_s = \psi_s / \Sigma_s\ \psi_s = \psi_s / p_f = (1+r_f)\ \psi_s,\ s=1,\ldots,S\}$
    - $q_s > 0$
    - $\Sigma_s\ q_s = 1$
- Risk neutral valuation of these assets
  - $p_i = \Sigma_s\ \psi_s\, v_{is} = p_f\, \Sigma_s\, (\psi_s / p_f)\, v_{is}$
    $= (1+r_f)^{-1}\, \Sigma_s\, q_s\, v_{is} = E_q[v_i]/(1+r_f)$
  - $E_q[\cdot]$ = expectation w.r.t. risk neutral measure

Risk Neutral Probabilities with No Riskless Asset

- Suppose N+1st asset has limited liability
  - i.e., $v_{N+1s} > 0 \; \forall \; s$
- Use N+1st asset as the numeraire
  - $p_i^* = p_i/p_{N+1}$; $v_{is}^* = v_{is}/v_{N+1s}$
  - $\Rightarrow$ N+1st asset is riskless in * numeraire
    - i.e., $v_{N+1,s}^* = v_{is}/v_{N+1s} = 1 \; \forall \; s$
  - $p_{N+1}^* = 1 \Rightarrow r_f = 0$ in * numeraire with = 0
- Risk neutral pricing
  - $p_i^* = \Sigma_s \psi_s v_{is}^* \Rightarrow p_i = \kappa \Sigma_s q_s v_{is}$; $\kappa = \Sigma_s \psi_s p_{N+1}/v_{N+1s}$
  - $q_s = \psi_s p_{N+1}/\kappa v_{N+1s}$

The Change-of-Measure Interpretation of SDF Pricing

- SDFs and risk neutral probabilities
  - $m_s \equiv \psi_s/\pi_s = (1 + r_f)\ \psi_s/[(1 + r_f)\ \pi_s]$
    $= (q_s/\pi_s)/(1 + r_f) \equiv \ell_s/(1 + r_f)$
  - ℓ_s:
    - Likelihood ratio for risk neutral vs. given probabilities
    - Radon-Nikodym derivative of q w. r. t. π
    - $E_\pi[\ell] = \Sigma_s\ \pi_s\ \ell_s = \Sigma_s\ \pi_s\ q_s/\pi_s = \Sigma_s\ q_s = 1$
- Revised SDF valuation relation
  - $p_i = \Sigma_s\ \pi_s\ v_{is}\ [(q_s/\pi_s)/(1 + r_f)]$

The Varieties of Martingales in the Various Pricing Relations

- Martingales under π
  - $E_\pi[\ell] - 1 = [\Sigma_s\ \pi_s\ q_s/\pi_s] - 1 = \Sigma_s\ q_s - 1 = 0$
  - $[(1 + r_f)\ E_\pi(m)] - 1 = E_\pi[\ell] - 1 = 0$
  - $E_\pi(R_i m) - 1 = \Sigma_s\ (\pi_s \psi_s/\pi_s) R_{is} - 1 = \Sigma_s\ \psi_s R_{is} - 1 = 0$
- Martingales under q
  - $E_q[R_i] - (1 + r_f) = 0$
- Transformed martingales
  - $R_f \times E_\pi(R_i m) = E_\pi(R_i \ell) = \Sigma_s\ \pi_s\ (q_s/\pi_s)\ R_{is} = E_q[R_i]$
    - Called Girsanov's theorem in diffusion models

Portfolio Choice and Pricing Relevant States

- Suppose investor j holds portfolio ω_j of these assets for some reason
- Portfolio payoffs: $v_{\omega_j} = \{v_{\omega_j s}, s = 1,\ldots,S\}$
- $\Rightarrow$ Payoff relevant states for *this* investor are those elements of v_{ω_j} that differ
  - i.e., a *coarser* partition than original S states
- $\Rightarrow$ Any economic theory that implies that some investor holds a particular portfolio is a theory of *pricing relevant states*

Some Economics for Pricing Relevant States

- Risk pooling
  - Stochastic process for returns in this market permit diversification
  - Investors who hold *any* diversified portfolio in this market are like investor j
- Risk sharing
  - With similar and sufficiently independent budget constraints, *all* investors can find it optimal to hold particular portfolios because of:
    - identical preferences
    - particular return distributions

An Abstract Definition of Pricing Relevant States

- What is a pricing relevant state?
  - Original state space {s=1,…,S}
  - SDF: $\{m_s = \psi_s/\pi_s, s=1,\ldots,S\}$
  - A pricing relevant state consists of all states with same state price per unit probability
  - i.e., state m = $\{s \in S: \psi_s/\pi_s = \psi_m/\pi_m\}$
- $\Rightarrow$ M pricing relevant states {m=1,…,M}
  - Partition S states into M groups with S_m states in each group such that $\Sigma_m S_m = S$

Indicators of the Payoff/Pricing Relevant State Distinction

- Indicators of events in S and M
  - 1_m = indicator of event $m \in M$
    - $\psi_m = \psi(1_m)$ = price of claim that pays \$1 if m occurs
  - 1_s = indicator of event $s \in S$
    - $\psi_s = \psi(1_s)$ = price of claim that pays \$1 if s occurs
- Indicators of events in S_m
  - $1_{ms} = 1_m 1_s = 1_m 1_{s \in S_m}$ = indicator of event $ms \in S_m$
    - $\psi_{ms} = \psi(1_m 1_s) = \psi(1_m 1_{s \in S_m})$ = price of claim that pays \$1 if both m and $s \in S_m$ occur

Bayes' Rule and Probabilities and Pricing on Payoff Relevant States

- Bayes' rule and probabilities of payoff and pricing relevant states
  - $\pi_{ms} = \pi_{s|m}\,\pi_m$
- Bayes' rule and pricing on payoff relevant states $s \in S_m$
  - Assertion that claims ms, $s \in S_m$ are priced *risk-neutrally* given that state m occurs
  - $\psi_{ms}/\pi_{ms} = \psi_m/\pi_m \Rightarrow \psi_{ms}/(\pi_{s|m}\pi_m) = \psi_m/\pi_m$
    $\Rightarrow \psi_{ms} = \pi_{s|m}\,\psi_m$
  - $\Rightarrow \psi_{ms} = \psi(1_m 1_{s \in S}) = \psi(1_m 1_{s \in S_m}) = \pi_{s|m}\,\psi_m$

Implications of Pricing/Payoff Relevant State Distinction

- Implicit assertion: Claims ms, $s \in S_m$, that:
  - investors view as perfect substitutes
    - Claims ms provide same insurance, i.e., hedges
  - should therefore sell for same price adjusted for probability of occurrence of substate ms
    - ms is *idiosyncratic* risk that should be *unpriced*
  - $\Rightarrow$ risk-neutral price given that state m occurs
- $\Rightarrow$ returns within pricing relevant states are perfect substitutes empirically
  - Pooling within-state data serves to refine state price density estimates in that state

Example: The Pricing of Pure Endowment Insurance

- Suppose the asset menu consists of:
  - Pure discount bonds of all maturities
  - What is called by actuaries t-period pure endowment insurance: a claim that pays $1 if the insured is alive at date t and zero otherwise
    - This is a digital option written on life expectancy
      - a non-price state variable
    - pure Arrow-Debreu mortality claim = bull spread in digital options struck at mortality dates t and t+1
      - pays $1 if the insured dies between t and t+1 and zero otherwise =
- How should pure endowments be priced?

Example: The Pricing of Pure Endowment Insurance, Continued

- Let $\psi_i(1_{T_i^{\mathscr{M}} \geq t})$ denote the price of a claim that pays \$1 if insured lives at least t periods
  - $T_i^{\mathscr{M}} \leq T_i^{\bar{\mathscr{M}}}$ is the date of death of individual i
- Price of a claim that pays \$1 if individual dies in year t and zero otherwise is:
  - $\psi_i(1_{T_i^{\mathscr{M}} = t}) = \psi_i(1_{T_i^{\mathscr{M}} \geq t}) - \psi_i(1_{T_i^{\mathscr{M}} \geq t+1})$
- What if endowments priced risk neutrally independent of stochastic interest rates?
  - $\psi_0(1_{T_i^{\mathscr{M}} \geq t}) = P_0(t)E[1_{T_i^{\mathscr{M}} \geq t} \mid \mathscr{F}_0] = \dfrac{1}{[1+y_0(t)]^t} E[1_{T_i^{\mathscr{M}} \geq t} \mid \mathscr{F}_0]$

Example: The Pricing of Pure Endowment Insurance, Continued

- Pure mortality contingent claim prices under risk neutral pricing of mortality risk
  - $\psi_i(1_{T_i^{\mathscr{M}}=t}) = \psi_i(1_{T_i^{\mathscr{M}}\geq t}) - \psi_i(1_{T_i^{\mathscr{M}}\geq t+1})$
    $= P_0(t)E[1_{T_i^{\mathscr{M}}\geq t} \mid \mathscr{F}_0] - P_0(t+1)E[1_{T_i^{\mathscr{M}}\geq t+1} \mid \mathscr{F}_0]$
- Payoff vs. pricing relevant states
  - Payoff relevant states are possible future paths of interest rates
  - Actuarially fair endowment insurance pricing means payoff relevant states associated with mortality contingent claims priced risk neutrally

Some Arithmetic for Aggregate Wealth

- The market portfolio of all risky assets
  - Portfolio weights: $w_{im} = MV_i/MV$
    - MV_i = # of shares of i outstanding x P_i
    - $MV = \sum_i MV_i$
- Suppose market cannot die
  - i.e., $Pr(R_m=0) = 0 \Rightarrow R_{ms} > 0 \; \forall \; s$
  - $\Rightarrow$ m optimal for some risk averse investor
  - Limited liability is sufficient but not necessary
    - i.e., $Pr(R_i=0 \; \forall \; i) = 0 \Rightarrow R_{is} \geq 0 \; \forall \; i, s$
    - precludes joint normality

Pricing Relevant States and Mean-Variance Efficiency

- Let m be efficient orthogonal partner of f
  - $E(R_i - R_f) = \beta_i E(R_m - R_f) \equiv \beta_i \lambda_m$
    - $\Rightarrow R_i - R_f = \beta_i (R_m - R_f) + \varepsilon_i; E[\varepsilon_i(R_m - R_f)] = 0$
- Pricing relevant states: $\{R_{ms}, ms=1,\ldots,\mathcal{M}\}$
  - $\Rightarrow \sum_s \pi_s (R_{is} - R_f) = \beta_i\lambda_m = \sum_s \pi_s [R_{is}(R_{ms} - \mathcal{R}_m)] \lambda_m$
  - $\Rightarrow \sum_s \pi_s R_{is}[1 - \lambda_m(R_{ms} - \mathcal{R}_m)] = R_f$
- $\Rightarrow m_s = \psi_s/\pi_s = [1-\lambda_m(R_{ms}-\mathcal{R}_m)]/R_f$ is SDF iff $[1-\lambda_m(R_{ms}-\mathcal{R}_m)] > 0$
  - $\Rightarrow q_s = \pi_s [1-\lambda_m(R_{ms}-\mathcal{R}_m)]$
  - $\psi_{ms} = \pi_{s|m}\, \psi_m$ as before

The Payoff/Pricing Relevant Distinction and Asset Menus

- As asset menu expands (i.e, as N grows):
  - Payoff relevant state space (i.e., S) grows
    - Magnitudes of stochastic discount factors and state prices generally change
  - Pricing relevant state space unchanged if additional assets priced on these states
    - i.e., M remains constant $\forall$ i s. t. $\psi_S(v_i) = \psi_M(v_i)$
- $\Rightarrow$ Elementary contingent claims written on the pricing relevant state space are implicit building blocks of prices of all such assets

Payoff and Pricing Relevance in a Simple Factor Pricing Model

- Suppose each firm payoff is given by:
  - $v_i = v_m + \varepsilon_i$; $E[\varepsilon_i | v_m] = 0$
- Suppose ε_i is Rothschild-Stiglitz noise:
  - i.e., $\psi(\varepsilon_i) = \Sigma_m \Sigma_{s \in S_i} \psi_{ms} \varepsilon_{is} = \Sigma_m \psi_m \Sigma_{s \in S_i} \pi_{s|m} \varepsilon_{is} = 0$
  - $\Rightarrow$ Investors want to eliminate ε_i risk from their portfolios if their utilities are independent of ε_i
- Then ε_i is unpriced by assumption and the price of this asset is given by:
  - $\Rightarrow p_i = \psi(v_i) = \Sigma_m \Sigma_{s \in S_i} \psi_{ms} [v_m + \varepsilon_{is}]$
    $= \Sigma_m \psi_m v_m + \Sigma_{s \in S_i} \pi_{s|m} \varepsilon_{is} = \Sigma_m \psi_m v_m$

Diversification in the Simple Factor Pricing Model

- Now suppose ε_i is idiosyncratic to asset i
  - $E[\varepsilon_i^2|\nu_m] = \sigma_{im}^2 < \sigma^2$; $E[\varepsilon_i\varepsilon_j|\nu_m] = 0 \;\forall\; i \neq j$
  - $\Rightarrow \varepsilon_i$ is diversifiable by WLLN:
    - $\Sigma_i\, w_i^2 \to 0 \Rightarrow \Sigma_i\, w_i\varepsilon_i \to 0$ since $\sigma(w)^2 \leq \Sigma_i\, w_i^2\, \sigma^2 \to 0$
- If the utility functions of investors:
  - are independent of pricing irrelevant ε_i risk, diversification lets them eliminate it in the limit
  - depend on ε_i risk, they will want to be long more of i if marginal utility is decreasing in it and long less or short if marginal utility is increasing in ε_i

Large Asset Menus and Mean/Variance Analysis

- N indexes number of assets in R_N, $\mathcal{R}_N$, V_N
- Efficient set parameters same as in Roll
  - $a_N = \mathcal{R}_N' V_N^{-1} \mathcal{R}_N$; $b_N = \mathcal{R}_N' V_N^{-1} \iota_N$; $c_N = \iota_N' V_N^{-1} \iota_N$
- Minimum variance portfolio parameters
  - $\mathcal{R}_{0N} = b_N/c_N$; $\sigma_{0N}^2 = 1/c_N$; $w_{0N} = V_N^{-1} \iota_N / c_N$
- Maximum squared Sharpe ratio portfolio parameters
  - $\mathcal{R}_{\mathcal{S}N} = a_N/b_N$; $\sigma_{\mathcal{S}N}^2 = a_N/b_N^2$; $w_{\mathcal{S}N} = V_N^{-1} \mathcal{R}_N / b_N$

The Minimum Variance Portfolio and the Absence of Arbitrage

- Theorem: $\lim_{N\to\infty} a_N - b_N \mathcal{R}_{0N} < \infty$
- Proof:
  - $x_N = b_N(w_{sN} - w_{0N})$
    - $x_N'\iota = 0$; $\mathcal{R}_{xN} = \sigma_{xN}^2 = a_N - b_N\mathcal{R}_{0N}$
  - $x_{N*} = x_N/(a_N - b_N\mathcal{R}_{0N})$
    - $\mathcal{R}_{xN*} = 1$; $\sigma_{xN*}^2 = (a_N - b_N\mathcal{R}_{0N})^{-1}$
  - $\lim_{N\to\infty}(a_N - b_N\mathcal{R}_{0N})^{-1} > 0 \Rightarrow \lim_{N\to\infty} a_N - b_N\mathcal{R}_{0N} < \infty$
  - Simply second order condition for unique solution to minimum variance program
  - i.e., nonuniqueness $\Rightarrow$ multiple riskless rates

The Limiting Behavior of σ_0^2

- Two cases
  - $c_N \to \infty \Rightarrow \sigma_{0N}^2 \to 0$
    - i.e., minimum variance portfolio is riskless
  - $c_N \to c_\infty < \infty \Rightarrow \sigma_{0N}^2 \to c_\infty^{-1} > 0$
- Theorem: Consider sequence $|V_N| > 0$. Then there is no riskless portfolio of risky assets iff $\exists\ \alpha_N \neq 0\ \forall$ N s. t.:
  - $V_N = \iota_N \iota_N' \alpha_N^2 + \Omega_N, |\Omega_N| \geq 0$
    - Intuition: Cannot eliminate equicorrelated component but can eliminate α risk if loadings differ from unity

The Limiting Behavior of σ_0^2, Continued

- Proof: Recall $w_{0N} = V_N^{-1}\iota_N/c_N$ and consider population projection of R_N on R_{0N}:
  - $R_N = \delta_N + [\mathrm{Cov}(R_N,R_{0N})/\sigma_{0N}^2]\, R_{0N} + \varepsilon_{0N}$
    $= \delta_N + V_N V_N^{-1}\, \iota_N\, (\sigma_{0N}^2/\sigma_{0N}^2)\, R_{0N} + \varepsilon_{0N}$
    $= \delta_N + \iota_N R_{0N} + \varepsilon_{0N}$
  - $\Rightarrow V_N = \iota_N \iota_N' \alpha_N^2 + \Omega_N,\ \Omega_N = E[\varepsilon_{0N}\varepsilon_{0N}'], |\Omega_N| \geq 0$
    - i.e., decomposition always exists with $\sigma_{0N}^2 = \alpha_N^2$
  - $\lim_{N\to\infty} \alpha_N^2 > 0 \Rightarrow w_N' V_N w_N = \alpha_N^2 + w_N'\Omega_N w_N$
    $> \alpha_N^2 > 0\ \forall\ w_N$
  - $\lim_{N\to\infty} \alpha_N^2 = 0 \Rightarrow \lim_{N\to\infty} \sigma_{0N}^2 = 0$

The Limiting Efficient Frontier with a Limiting Riskless Asset

- Limiting second order condition:
  - $c_N \to \infty$ and $a_N - b_N \mathcal{R}_{0N} \to k < \infty \Rightarrow a_N/c_N \to \mathcal{R}_{0N}{}^2$
- Limiting behavior of frontier variances
  - $\sigma_{pN}{}^2 = (a_N - 2b_N \mathcal{R}_p + c_N \mathcal{R}_p{}^2)/(a_N c_N - b_N{}^2)$
    $\to [\mathcal{R}_{0N}{}^2 - 2\mathcal{R}_p \mathcal{R}_{0N} + \mathcal{R}_p{}^2]/(a_N - b_N \mathcal{R}_{0N})$
    $\to (\mathcal{R}_p - \mathcal{R}_{0N})^2/(a_N - b_N \mathcal{R}_{0N})$
    $\to (\mathcal{R}_p - \mathcal{R}_f)^2/(a_\infty - b_\infty \mathcal{R}_f)$
- Limiting zero beta portfolio behavior:
  - $\mathcal{R}_{zN} = (a_N - b_N \mathcal{R}_p)/(b_N - c_N \mathcal{R}_p)$
    $= (\mathcal{R}_{0N}{}^2 - \mathcal{R}_{0N} \mathcal{R}_p)/(\mathcal{R}_{0N} - \mathcal{R}_p) \to \mathcal{R}_f$

Linear Factor Pricing Models and Diversification

- Linear factor model for returns
  - $\tilde{R}_N = \mathcal{R}_N + B_N\tilde{\delta} + \tilde{\varepsilon}_N$; $E(\tilde{\varepsilon}_N) = E(\tilde{\delta}\tilde{\varepsilon}_N) = 0$; $\Omega_N = E[\tilde{\varepsilon}_N\tilde{\varepsilon}_N']$
  - Assume $\tilde{\varepsilon}_N$ is diversifiable risk
    - $\lim_{N\to\infty} \xi_{max}(\Omega_N) < \infty$ (i.e., a WLLN holds)
      - $\xi_{max}(\bullet)$ = largest eigenvalue of its argument
- Well-diversified portfolios
  - w_{PN} s. t. $w_{PN}'\iota = 1$ and $\lim_{N\to\infty} w_{PN}'w_{PN} = 0$
  - $\tilde{R}_{PN} = w_{PN}'\tilde{R} = \beta_p'\tilde{\delta} + w_{PN}'\tilde{\varepsilon}_N$
  - $\lim_{N\to\infty}\sigma_{\varepsilon_{PN}}^{\ 2} = Var(w_{PN}'\tilde{\varepsilon}_N) = \lim_{N\to\infty} w_{PN}'\Omega_N w_{PN}$
    $\leq \lim_{N\to\infty} w_{PN}'w_{PN}\ \xi_{max}(\Omega_N) = 0$

The Intuition Behind the Arbitrage Pricing Theory (APT)

- All well-diversified portfolios have linearly dependent returns in the limit
  - $\lim_{N\to\infty}[\tilde{R}_{PN} - \mathcal{R}_N] = \lim_{N\to\infty}[\beta_p'\tilde{\delta} + w_{PN}'\tilde{\varepsilon}_N] = \beta_p'\tilde{\delta}$ $\Rightarrow$ at most K imperfectly correlated well-diversified portfolios
- Linearly dependent portfolio returns

  $\Rightarrow$ collinear expected returns
  - $\lim_{N\to\infty}\tilde{R}_{pN} = \mathcal{R}_N + \beta_p'\tilde{\delta} \Rightarrow \lim_{N\to\infty}\mathcal{R}_N = \lambda_0 + \beta_p'\lambda$
- $\Rightarrow$ (sort of) collinear expected individual asset returns

Costless Zero Factor Beta Portfolios

- Consider population GLS projection of expected returns $\mathcal{R}_i$ on 1 and β_i
  - $\mathcal{R}_i = \tau_{0N} + \beta_i' \tau_N + \alpha_{iN}$; $\alpha_N'\Omega_N^{-1}\iota = 0$

    $\alpha_N'\Omega_N^{-1}B_N = 0$
  - $\Rightarrow \Omega_N^{-1}\alpha_N$ is a zero net investment, zero factor risk portfolio
- Scaling by $(\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}$ preserves zero cost and factor risk nature of portfolio
  - i.e., $z_N = (\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}\Omega_N^{-1}\alpha_N$ is a zero net investment, zero factor risk portfolio

The Arbitrage Pricing Theory

- Moments of z_N payoff
  - $\mathcal{R}_{z_N} = (\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}\alpha_N'\Omega_N^{-1}\mathcal{R}$
    $= (\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}(\alpha_N'\Omega_N^{-1}\alpha_N) = 1$
  - $\sigma_{z_N}^{\ 2} = \mathrm{Var}[(\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}\alpha_N'\Omega_{\varepsilon_N}^{-1}\tilde{\mathrm{R}}]$
    $= \mathrm{Var}[(\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}\alpha_N'\Omega_N^{-1}\tilde{\varepsilon}_N]$
    $= (\alpha_N'\Omega_N^{-1}\alpha_N)^{-1}$
  - $\Rightarrow z_N'\tilde{\mathrm{R}}$ is an arbitrage unless $\alpha_N'\Omega_N^{-1}\alpha_N < \infty$
- $\Rightarrow \mathcal{R}_i \approx \tau_{0N} + \beta_i'\tau_N$
  - i.e., expected returns of assets in menu are linearly dependent in the limit in an ε, δ sense
    - WLS version bounds $\alpha_N'A\alpha_N$ $\forall$ $|A| > 0$

The Arithmetic of Factor Risk Premiums and Factor Pricing

- Let pN be an arbitrary efficient portfolio
- Asset betas with respect to pN
  - $\beta_{pN} = V_N w_{pN}/\sigma_{pN}^2 = [B_N V(\tilde{\delta}_x) B_N' + \Omega_N] w_{pN}/\sigma_{pN}^2$
    $\equiv B_N \gamma_{pN}/\sigma_{pN}^2 + \Omega_N w_{pN}/\sigma_{pN}^2;\ \gamma_{pN} = V(\tilde{\delta}_x) B_N' w_{pN}$
- Factor risk premiums
  - $\mathcal{R}_i = \mathcal{R}_{pz_N} + \beta_{ipN}(\mathcal{R}_p - \mathcal{R}_{pz_N})$
    $= \mathcal{R}_{pz_N} + \beta_i' \gamma_{pN}(\mathcal{R}_p - \mathcal{R}_{pz_N})/\sigma_{pN}^2$
    $+ \Omega_N w_{pN}(\mathcal{R}_p - \mathcal{R}_{pz_N})/\sigma_{pN}^2$
    $= \mathcal{R}_{pz_N} + \beta_i' \gamma_{pN}(\mathcal{R}_p - \mathcal{R}_{pz_N})/\sigma_{pN}^2 + \alpha_{ipN}$
    - zN is orthogonal partner of (i.e., zero beta portfolio wrt) pN

Diversification and Factor Risk Premiums

- Pricing errors and the efficient frontier slope
  - $\alpha_{pN}'\Omega_N^{-1}\alpha_{pN} = [(\mathcal{R}_p - \mathcal{R}_{pz_N})^2/\sigma_{pN}^4]$
    $\times\ w_{pN}'\Omega_N\Omega_N^{-1}\Omega_N w_{pN}$
    $= [(\mathcal{R}_p - \mathcal{R}_{pz_N})^2/\sigma_{pN}^4] w_{pN}'\Omega_N w_{pN}$
    $= [(\mathcal{R}_p - \mathcal{R}_{pz_N})^2/\sigma_{pN}^2][1 - R_{pN\delta}^2]$
  - $1 - R_{pN\delta}^2 = w_{pN}'\Omega_N w_{pN}/w_{pN}'V_N w_{pN} = \sigma_{\varepsilon_{pN}}^2/\sigma_{pN}^2$
- $\Rightarrow$ There is exact factor pricing iff $\exists$ a limiting well-diversified efficient portfolio p
  - $\lim_{N\to\infty} \sigma_{\varepsilon_{pN}}^2 \le \lim_{N\to\infty} w_{pN}'w_{pN}\, \xi_{max}(\Omega_N) = 0$ iff p is well-diversified

Factor Risk Premiums and the Limiting Efficient Frontier

- Factor pricing with a limiting riskless asset
  - $\mathcal{R}_{pz_N} \rightarrow \mathcal{R}_f \Rightarrow \tau_{0N} \rightarrow \mathcal{R}_f$
    $\Rightarrow \tau_N \rightarrow \gamma_p(\mathcal{R}_p - \mathcal{R}_f)/\sigma_p^2$
  - i.e., $\gamma_{pN} = V(\tilde{\delta}_x)B_N' w_{pN} \rightarrow \gamma_p$
- Factor pricing when $\sigma_{0N}^2 > \sigma_0^2 > 0$
  - Set $w_{pN} = w_{\mathcal{S}N} = V_N^{-1}\mathcal{R}_N (\iota_N' V_N^{-1} \mathcal{R}_N)^{-1}$
    - Maximum squared Sharpe ratio portfolio
  - $\Rightarrow \tau_{0N} = 0 =$ zero beta rate of portfolio $\mathcal{S}$
  - $\tau_N = \gamma_{\mathcal{S}N}\mathcal{R}_{\mathcal{S}N}/\sigma_{\mathcal{S}N}^2 \Rightarrow \tau_N \rightarrow \gamma_{\mathcal{S}}\mathcal{R}_{\mathcal{S}}/\sigma_{\mathcal{S}}^2$
  - i.e., $\gamma_{\mathcal{S}N} = V(\tilde{\delta})B_N' w_{\mathcal{S}N} \rightarrow \gamma_{\mathcal{S}}$

Why Factor Pricing Models Are Economically Important

- Two distinct notions of nonsystematic or idiosyncratic risk
  - Risk for which investors receive no premium
  - Risk that can be eliminated via diversification
- Investors and investment textbooks (but not necessarily all finance professors) have strong priors that diversifiable risk can be identified and is 'not priced'
- If true, linear asset pricing models must be based on well-diversified portfolios