Week Two

Tautologies and Economics in the Interpretation of the Fundamental Theorem of Asset Pricing

Some of the Economic Issues in the Fundamental Theorem

- Incomplete markets and the set of state prices
- Options and spanning
- Modigliani-Miller and value additivity
- Numeraires
- A digression on asymmetric information

The Economics of Spanning for Payoff Relevant States

- Suppose:
  - *payoff relevant* state s not spanned
    - i.e., cannot insure against occurrence of s
      - $\nexists\ \omega$ s. t. $\mathcal{V}'\omega = e_s$
  - at least two investors have different shadow prices for claim e_s
- $\Rightarrow$ Financial engineer earns arbitrage profit by selling it to high valuation investor and buying claim e_s from low valuation investor
  - Competition among engineers should eliminate any rents in equilibrium

The Economics of Spanning for Payoff Relevant States, Continued

- No barriers to free supply of butterflies
  - No lack of observability or moral hazard to complicate the writing of derivative contracts
  - No frictions makes it feasible
  - Profit motive provides incentive
- $\Rightarrow$ Natural presumption: markets generically complete in a payoff relevant sense
  - Unless frictions are important

Butterfly Prices and the State Price Density

- Butterfly price
  - $p_b = [C(p,v_s-\Delta v) - 2C(p,v_s) + C(p, v_s+\Delta v)]/\Delta v$
    $= \{[C(p,v_s-\Delta v) - 2C(p,v_s) + C(p,v_s+\Delta v)]/(\Delta v)^2\}\Delta v$
- Limiting butterfly prices
  - $\Rightarrow \lim_{\Delta v \to 0} p_b(v_s) = \{\partial^2 C(p,E)/\partial E^2\}|_{E=v_s}\, dv$ is the state price density
  - $\Rightarrow$ State price density is C2
- Limiting price of arbitrary derivatives
  - $p_{f(v)} = \int f(v)\, p_b(v) = \int f(v)\, [\partial^2 C(p,v)/\partial v^2]\, dv$

Alternative Numeraires: The Nominal/Real Distinction

- Traded assets typically denominated in units of account
- Absence of arbitrage a property of price system irrespective of numeraire
  - could base real returns on:
    - some price deflator (popular empirically)
    - p_f (popular in general equilibrium finance)
- Absence of arbitrage restricts nominal asset prices because investors prefer more nominal wealth to less

Alternative Numeraires: International Arbitrage Pricing

- Changes of numeraire arise naturally in an international setting
- Positive state prices must exist in every currency of denomination so long as foreign exchange markets satisfy the *Perfect Market Assumptions*
  - $\{\psi_{\text{foreign,s}}, \text{s}=1,\ldots,\text{S}\} = \{e_{\text{s}}\ \psi_{\text{domestic,s}}, \text{s}=1,\ldots,\text{S}\}$
    - e_{s} = foreign/domestic currency
    - State space must be augmented to accommodate exchange rate states if necessary

Heterogeneous Information and State Prices

- Reasonable suppositions
  - Some investors better informed than others
  - Many (most?) investors better informed than econometric students of asset markets
- Implications
  - Refine payoff relevant state s into states with differing information substates
    - termed finer state partitions
  - (Some) investors might have more accurate probability estimates than others

Asymmetric Information and No-Arbitrage Pricing

- Prices, not probabilities, are the issue
- Investors with superior information still must perceive no arbitrage opportunities
  - An investor j who can:
    - trade without frictions in these assets
    - partition state s into disjoint K *possible* substates
  - must perceive no arbitrage possibilities so:
    - $\psi_s = \Sigma_k \ \psi_{jsk}$
      - ψ_{jsk} = investor j's perceived state price for substate k
      - ψ_s = arbitrage-free payoff relevant state price
      - Only one such price if this market is spanned

The Fundamental Theorem of Asset Pricing Theory Again

- *The Fundamental Theorem of Asset Pricing:* Given assumptions 1-3, a positive pricing rule $\psi(\bullet)$ supported by a set of positive state prices $\{\psi_s = \psi(e_s), s=1,\ldots,S\}$, not necessarily unique, exists iff there are no strong arbitrage opportunities (i.e., possible free lunches) in this market
  - $\Rightarrow p_i = \Sigma_{s \in S}\ \psi_s v_{is}$
    - e_s = s-vector of zeros except for a one in position s

The Fundamental Theorem, Alternative Proof

- Alternative proof strategy (with added economic content)
  - Absence of arbitrage $\Rightarrow$ existence of interior solution to portfolio optimization problem of investor trading in these assets
  - Finite solution to maximum problem $\Rightarrow$ existence of positive linear pricing rule
  - Already proved existence of positive linear pricing rule $\Rightarrow$ no-arbitrage
    - $\omega' \mathrm{p} = \omega' \mathcal{V} \psi$ and $\omega' \mathcal{V} \geqq 0 \Rightarrow \omega' \mathrm{p} > 0$ because $\psi > 0$

The Fundamental Theorem, Alternative Proof, Continued

- Consider a (fictitious) investor who:
  - prefers more to less
  - maximizes a general, differentiable two period utility function
    - nondifferentiable utility can be handled at some cost
  - invests only in these assets
- Take initial (optimal) consumption as given to reduce to single period portfolio problem
  - $\Rightarrow$ Lagrange multiplier of wealth constraint is marginal utility of first period consumption

The Fundamental Theorem, Alternative Proof, Continued

- (Fictitious) portfolio choice problem:
  - $\max_{\omega} u[\Sigma_i\ \omega_i v_{i1},\ldots,\Sigma_i\ \omega_i v_{iS}]$ s. t. $W_0 = \Sigma_i\ \omega_i\, p_i$
  - note that:
    - investor's (arbitrary) probabilities implicit in u[•]
    - invested wealth W_0 and $\{p_i;\ i=1,\ldots,N\}$ in real terms
      - role of numeraire sometimes nontrivial in general equilibrium theory
      - convenient choice is $p_f = R_f^{-1}$ if available
- $\mathcal{L} = u[\Sigma_i\ \omega_i v_{i1},\ldots,\Sigma_i\ \omega_i v_{iS}] + \mu[W_0 - \Sigma_i\ \omega_i p_i]$
  - $\Rightarrow \Sigma_{s\in S}\, u_s[W_1^*,\ldots,W_S^*]\, v_{is} = \mu\, p_i$
  - μ = marginal utility of first period consumption

The Fundamental Theorem, Alternative Proof, Continued

- Maximum problem has no interior solution if there is an arbitrage opportunity
  - End-of-period wealth W_s^* is unbounded in at least one state if there is an arbitrage opportunity
  - Wealth in other states can be increased without bound by shifting investment out of state s* to the other states since W_s^* remains unbounded
- Interior maximum and differentiable utility $\Rightarrow$ FONCs characterize portfolio choice

The Fundamental Theorem, Alternative Proof, Continued

- Euler equations of this hypothetical investor deliver the linear pricing rule
  - $\Sigma_{s \in S} u_s[W_1^*, \ldots, W_S^*] \, v_{is} = \mu \, p_i \; (= E\{u_c[c_0, \tilde{W}]\} \; p_i)$
  - $\Rightarrow p_i = (1/\mu) \, \Sigma_{s \in S} \, u_s[W_1^*, \ldots, W_S^*] \, v_{is}$
  - $\Rightarrow p_i = \Sigma_{s \in S} \; \psi_s \, v_{is}; \;\; \psi_s = u_s[W_1^*, \ldots, W_S^*]/\mu > 0$
- Note generality of construction
  - Arbitrary (non-expected utility) preferences
  - Arbitrary (implicit) probability beliefs
  - Arbitrary dependence of choice problem on other state variables

State Dependent Expected Utility and the Fundamental Theorem

- (Fictitious) portfolio choice problem
  - $\max_{\omega} \Sigma_s\ \pi_s\ u[\Sigma_i\ \omega_i v_{is}, s]$ s. t. $W_0 = \Sigma_i\ \omega_i\ p_i$
  - $\Rightarrow \mathcal{L} = \Sigma_s\ \pi_s\ u[\Sigma_i\ \omega_i v_{is}, s] + \mu[W_0 - \Sigma_i\ \omega_i\ p_i]$
    - $\{\pi_s > 0, s = 1, \ldots, S\}$ are arbitrary probabilities
- FONCs and the linear pricing rule
  - $\Sigma_s\ \pi_s\ u'[W_s^*, s]\ v_{is} = \mu\ p_i$
  - $\Rightarrow p_i = (1/\mu)\ \Sigma_s\ \pi_s\ u'[W_s^*, s]\ v_{is}$
  - $\Rightarrow p_i = \Sigma_s\ \psi_s\ v_{is};\ \ \psi_s = \pi_s\ u'[W_s^*, s]/\mu > 0$
- State prices are always consistent with state dependent expected utility maximizer

Any Portfolio Is ‘Efficient’ With State Dependent Preferences

- Endow an investor with arbitrary initial:
  - wealth and portfolio: $W_a = \Sigma_i \, p_i \, q_{ia}$
  - probability beliefs: $\{\pi_{as} = \pi_s > 0, s=1,\ldots,S)$
- Take same $u[\bullet,s]$ and arbitrary $v(\bullet)$ with $v' > 0$
- Set investor preferences to:
  - $U[\bullet,s] = v(\bullet)u'[W_s^*,s]/v'(W_{as}) \ \forall \ s$
  - $\Rightarrow U'[W_{as},s] = u'[W_s^*,s] \ \forall \ s$
- This investor holds portfolio *a* since:
  - $(1/\mu) \, \Sigma_s \, \pi_s \, U'[W_{as},s] \, v_{is} = (1/\mu) \, \Sigma_s \, \pi_s \, u'[W_s^*,s] \, v_{is}$

Heterogeneous Beliefs and State Dependent Expected Utility

- State dependence in utility observationally equivalent to heterogeneous beliefs
  - $\Sigma_s\ \pi_s\ u[\bullet,s] = \Sigma_s\ \pi_{as}\,(\pi_s/\pi_{as})\,u[\bullet,s]$
    $= \Sigma_s\ \pi_{as}\,u_a[\bullet,s];\ u_a[\bullet,s] = (\pi_s/\pi_{as})\,u[\bullet,s]$
  - $\Rightarrow p_i = (1/\mu)\ \Sigma_s\ \pi_s\,u'[\bullet,s]\ v_{is}$
    $= (1/\mu)\ \Sigma_s\ \pi_{as}\,(\pi_s/\pi_{as})\,u'[\bullet,s]\ v_{is}$
    $= (1/\mu)\ \Sigma_s\ \pi_{as}\,u_a'[\bullet,s]\ v_{is}$
- Originally state independent preferences of the form $\Sigma_s\ \pi_s\ u[\bullet]$ yield same portfolio as state dependent preferences $\Sigma_s\ \pi_{as}\ u_a[\bullet,s]$

A Simple Representative Investor Equilibrium Model of this Market

- Endow representative investor with:
  - Initial wealth $W_m = \Sigma_i \, p_i \, q_{im}$
  - Arbitrary probability beliefs $\{\pi_{ms} > 0, s=1,\ldots,S\}$
- Set arbitrary initial wealth W_a, $u[\bullet,s]$, and $v(\bullet)$, $v' > 0$ and solve for optimal state contingent wealth $W_s{*}$
- Representative investor chooses to hold aggregate wealth in this market since:
  - $U_m[\bullet,s] = v(\bullet)u[W_s{*},s]/v'(W_{ms})$
    $\Rightarrow \Sigma_s \, \pi_{ms} U_m'[W_{ms},s]v_{is} = \Sigma_s \, \pi_{ms} u'[W_s{*},s]v_{is}$

A Note on the Tautological Nature of Financial Economics

- What do excess demand curves show?
  - The Sonnenschein-Debreu-Mantel Theorem:
    - Loosely speaking, any set of *excess demand functions* can be rationalized as the choice of some optimizing consumer who satisfies WARP
  - Implication: the hypothesis that an economy is in equilibrium is empirically vacuous
- Finance version
  - Arbitrage-free asset *prices* can always be viewed as the result of a representative investor, rational expectations equilibrium

Why Asset Pricing Theory Is Vacuous in These Circumstances

- Arbitrage-free prices satisfy:
  - $p_i = \Sigma_s \ \psi_s v_{is} = \Sigma_s \pi_s m_s v_{is}$
    $= (1/\mu)\ \Sigma_s \ \pi_s U_m'[W_{ms},s]\ v_{is}$
- Unrestricted beliefs $\Rightarrow$ any $m_s > 0$ is consistent with some (set of) expectations
- For given beliefs, any $m_s > 0$ can be rationalized by some (set of) state dependent preferences via $m_s = U_m'[W_{ms},s]/\mu > 0$
  - Example: state dependent risk neutral utility
    - i.e., $U_m[W_{ms},s]/\mu = a_{ms}+b_{ms}W_{ms}>0$

Other Sources of State Dependencies in Preferences

- Portfolio choice part of larger economic life
  - Additional traded and nontraded wealth
  - Shocks to utility or habit formation/durability
  - Changing information/investment opportunities
  - Health and other idiosyncratic state variables
- $\Rightarrow$ rational trader with state independent von Neumann-Morgenstern preferences can have state dependent indirect utility for wealth
  - Only generically state independent with a lot of (perhaps unreasonable) separability

Implications of State Independent Investor Preferences

- Suppose investor j has state independent preferences $u_j(\bullet)$, beliefs π_{js}, and invests wealth W_{0j} only in these assets
  - $\max_{\omega} \Sigma_s\ \pi_{js}\ u_j[\Sigma_i\ \omega_{ij} v_{ijs}]$ s. t. $W_{0j} = \Sigma_i\ \omega_{ij}\ p_i$
- FONCs and the linear pricing rule
  - $\Sigma_s\ \pi_{js}\, u'[W_{js}]\ v_{is} = \mu_j\ p_i \Rightarrow \psi_{js} = \pi_{js} u'[W_{js}]/\mu_j$
  - $\Rightarrow u_j'[W_{js'}] \geq u_j'[W_{js}] \Leftrightarrow \psi_{js'}/\pi_{js'} \geq \psi_{js}/\pi_{js}$
- Investor j chooses optimal portfolios such that $W_{js'} \leq W_{js} \Leftrightarrow \psi_{js'}/\pi_{js'} \geq \psi_{js}/\pi_{js}$

Implications of State Independent Preferences with Common Beliefs

- Investor k with beliefs π_{js} like j but different utility $u_k(\bullet)$ form optimal portfolios s. t.:
  - $W_{ks'} \leq W_{ks} \Leftrightarrow u_k'[W_{ks'}] \geq u_k'[W_{ks}]$
- However, this need *not* imply:
  - $u_j'[W_{js'}] \geq u_j'[W_{js}] \Leftrightarrow u_k'[W_{ks'}] \geq u_k'[W_{ks}]$
- Reason: state prices generically not unique when S > N
  - $\psi_{js}/\pi_{js} \neq \psi_{ks}/\pi_{ks} \Rightarrow W_{js'} \leq W_{js} \not\Rightarrow W_{ks'} \leq W_{ks}$
  - State prices per unit probability generically lie in ordered N-dimensional subspace

The Introduction of Options in this Market

- What happens if options written on the assets in this market are introduced?
- Who knows? The payoff relevant state space changes and so:
  - New investors might be attracted to the market
  - Implicit state dependencies in preferences of existing investors might become explicit
- Question at hand is what investors participate in this market after options are introduced
  - N. B.: Short call options violate limited liability

A Model of the Role of Options in This Market

- After the introduction of options, suppose all investors in this market have:
  - common beliefs $\{\pi_s > 0, s=1,\ldots,S_D\}$
    - S_D is payoff relevant state space *inclusive* of options
  - state independent preferences $u_j(\bullet)$
  - invest all wealth only in these assets plus the options that have been introduced
- If options can be freely written on payoff relevant states space, the usual gains from trade argument $\Rightarrow$ market will be completed

Equilibrium Implications of Market Completion Via Options

- Optimal portfolios of all such investors s.t.:
  - $W_{js'} \leq W_{js} \Leftrightarrow u_j'[W_{js'}] \geq u_j'[W_{js}]$
    - Note wealth inclusive of option payoffs
- Completion of market $\Rightarrow$ unique state prices
  - $\Rightarrow \ \psi_{js}/\pi_{js} = \psi_s/\pi_s \ \forall$ j,s
  - $\Rightarrow u_j'[W_{js'}] \geq u_j'[W_{js}] \Leftrightarrow u_k'[W_{ks'}] \geq u_k'[W_{ks}]$
  - $\Rightarrow W_{js'} \leq W_{js} \Leftrightarrow W_{ks'} \leq W_{ks}$
- Optimal portfolios such that investor wealth across states has a *rank* correlation of one

Representative Investors and Efficient Set Convexity

- The set of optimal portfolios in completed market is convex under these assumptions:
  - Fix $\gamma \in (0,1) \Rightarrow \gamma W_{js'} + (1-\gamma) W_{ks'} \geq \gamma W_{js} + (1-\gamma) W_{ks}$
- Efficient set convexity $\Rightarrow$ market wealth s.t.:
  - $W_{ms'} \leq W_{ms} \Leftrightarrow \psi_{s'}/\pi_{s'} \geq \psi_s/\pi_s$
    - i.e., perfect rank correlation of investor wealth across states $\Rightarrow$ perfect rank correlation with market wealth
- $\exists$ exists a representative investor
  - $\exists\, u_m(\bullet)$ (not necessarily unique) with $u_m' > 0$ s.t.:
    - $u_m'[W_{ms'}] \geq u_m'[W_{ms}] \Leftrightarrow \psi_{s'}/\pi_{s'} \geq \psi_s/\pi_s \Leftrightarrow m_{s'} \geq m_s$

The Essence of the Capital Asset Pricing Model (CAPM)

- Implications of efficient set convexity
  - Options are in zero net supply so ∃ efficient portfolios with no option positions
  - Market portfolio must be held so one such portfolio must be the market
- All assets including options priced by:
  - $\Sigma_s \pi_s [u_m'(W_{ms})/\mu_m] v_{is} = p_i; i = 1,\ldots,N_D$
- Order states s. t. $\{W_{m1} \leq W_{m2} \leq \ldots \leq W_{mS}\}$
  - Pricing relevant state space is $\mathcal{M} \leq S$ distinct values of market wealth

Risk Exposures and Risk Premiums in *This* Market

- Investors agree on risk/return model
  - $\exists\, u_m(\bullet)$ s. t. $\Sigma_s\, \pi_s\, u_m'[W_{ms}]\, R_{is}/\mu_m \equiv \Sigma_s\, \pi_s m_s R_{is} = 1$
  - $R_i = \alpha_{i\pi} + \beta_{i\pi} m_\pi + \varepsilon_i$; $E_\pi[(\alpha_{i\pi} + \beta_{i\pi} m_\pi + \varepsilon_i)m] = 1$
    - $1 = E_\pi[(\alpha_{i\pi} + \beta_{i\pi} m_\pi + \varepsilon_i)m_\pi] = \alpha_{i\pi} R_f^{-1} + \beta_{i\pi}\, E_\pi(m^2)$
- $\Rightarrow R_i = R_f + \beta_{i\pi}\,[m_\pi - \lambda_\pi] + \varepsilon_i$; $E_\pi[\varepsilon_i] = 0$
  - $\lambda_\pi = R_f\, E_\pi(m^2) = R_f\,[R_f^{-2} + \sigma_{m\pi}^{\;2}] = R_f^{-1} + R_f\, \sigma_{m\pi}^{\;2}$
- Investors agree $\beta_{i\pi}$ measures risk exposure
  - Unsystematic risk ε_i earns no risk premium
- Investors agree λ_π is premium for m_π risk

The Economic Interpretation of an Empirical Search for $u_m(\bullet)$

- GE theory posits preferences and budget and other constraints and seeks (perhaps sets of) prices that clear markets
- In empirical work, one takes prices as given and seeks (perhaps sets of) preferences and budget constraints that explain prices
- $\Rightarrow$ Search for (perhaps sets of) $\{m_1, \ldots, m_{\mathcal{M}}\}$ or $u_m(\bullet)$ asks if a representative agent model can explain prices on this market, not the conditions under which such a model exists

The Economic Interpretation of a Successful Search for $u_m(\bullet)$

- Finding such pricing in different menus of traded assets would be puzzling in principle
  - Prices of original assets must remain in the span of their payoff relevant state space after more assets are added
  - 'Exchange rate' between original and additional assets must lie in the same span
- Exact separation of pricing on subsets implausible but 'approximate' separation can arise through diversification

Problems With the Assumption of Normally Distributed Payoffs

- Change model in one dimension: assume joint normality of primary asset payoffs
  - Inherently incompatible with limited liability
  - Roughly compatible with limited liability if:
    - $E[R_i] > 1$
    - $E[R_i]/\sigma(R_i) \gg 1$ for some i
  - Precisely compatible with limited liability in continuous time if mean and variance both of order dt $\Rightarrow$ prices follow a diffusion process
    - i.e., $\Rightarrow \lim_{dt\to 0} E[R_i]/\sigma(R_i)^2 = O(1)$ coupled with normality

A First Look at the Capital Asset Pricing Model (CAPM)

- Project R_i on R_m
  - $R_i = \alpha_{im} + \beta_{im} R_m + \varepsilon_{im}$; $E[\varepsilon_{im}] = E[R_m\ \varepsilon_{im}] = 0$
- Apply Euler equation
  - $E[(\alpha_{im} + \beta_{im}\ R_m + \varepsilon_{im}) u_m'(W_m)/\mu_m] = 1$
    - $E[\varepsilon_{im}\ u_m'(W_m)] = 0$ since ε_{im} and W_m are independent
    - $1 = \alpha_{im} R_f^{-1} + \beta_{im}\ E[R_m u_m'(W_m)/\mu_m] = \alpha_{im} R_f^{-1} + \beta_{im}$
      $\Rightarrow \alpha_{im} = [1 - \beta_{im}] R_f$
- $\Rightarrow$ Pricing of the normally distributed assets by the *Security Market Line*
  - $R_i = R_f + \beta_{im}[R_m - R_f] + \varepsilon_{im}$; $E[\varepsilon_{im}] = E[R_m \varepsilon_{im}] = 0$

The Risk Premium in This Model

- The Euler equation for the market portfolio
  - $E[R_m\, u_m'(W_m)/\mu_m] = 1$
- Stein's (and Rubinstein's) lemma
  - If x and y are jointly normal and f(•) is differentiable, $Cov[x,f(y)] = E[f'(y)]\, Cov[x,y]$
- Implied risk premium
  - $1 = E[R_m\, u_m'(W_m)/\mu_m]$
    $= E[R_m]E[u_m'(W_m)/\mu_m] + Cov[R_m,u_m'(W_m)]/\mu_m$
    $= E[R_m]\, R_f^{-1} + [W_0/\mu_m]E[u_m''(W_m)]\, Var[R_m]$
  - $\Rightarrow E[R_m - R_f] = -E[u_m''(W_m)]\, [R_f W_0/\mu_m]\, Var[R_m]$

Why This Is an Unconventional Route to the CAPM

- Asset menu
  - includes options
    - assets without normally distributed returns
  - not exhaustive
    - relies on spanning of a subset of assets with normally distributed returns, not on portfolio separation
- CAPM pricing relation:
  - relies on putative state independence of investor preferences in this market
  - does not apply to all returns
    - options do not lie on SML

Why This Is the Essence of the CAPM

- Market for primary assets completed over payoff relevant states
  - ⇒ state prices are unique
- Investors in this market
  - Invest only in assets in this market ⇒ investor portfolio returns lie in the same return space
  - Have state independent preferences ⇒ state dependent wealth is decreasing in state prices per unit probability for each marginal investor
    - i.e., state prices per unit probability can differ if investors have heterogeneous beliefs

Why This Is the Essence of the CAPM, Continued

- Investors in this market
  - Have homogeneous beliefs ⇒ state prices per unit probability are equated across investors
  - ⇒ each investor's wealth has a rank correlation of one that of all others in this market
  - ⇒ aggregate wealth over states also has a rank correlation of one with investor wealth
    - Efficient set wrt state independent preferences and homogeneous beliefs is convex
  - Market portfolio contains only primary assets
    - All assets priced by representative investor FONCs