

# The Theory and Testing of Intertemporal Asset Pricing Models

---

## Learning From Asset Prices in the Absence of Arbitrage

Economics 272

Department of Economics

University of California, San Diego

Bruce N. Lehmann

Graduate School of International Relations  
and Pacific Studies

University of California, San Diego

# The Two Flavors of Asset Pricing Theory

---

- Models in which asset prices reflect the absence of arbitrage opportunities
  - Example: The Modigliani-Miller debt and dividend irrelevance propositions
- Equilibrium models based on specific preferences and technology
  - Variations on abstract and elegant Arrow-Debreu general equilibrium model under uncertainty through valuation of state contingent claims implicit in traded assets

# Budget Constraint Economics

---

- Loosely speaking, an arbitrage involves making something for nothing
- All investors:
  - should try to exploit them vigorously with appropriate trading strategies
  - could increase their wealth by altering their portfolios to exploit any such opportunity
- The absence of arbitrage opportunities in financial markets is a necessary condition for equilibrium in frictionless markets

# The Absence of Arbitrage and Relations Among Asset Prices

---

- Arrow-Debreu prices exist in frictionless arbitrage-free markets
  - That is, pure state contingent claim prices
- Some portfolios of assets in such markets have perfectly correlated returns
  - Portfolio values determined by arbitrage considerations alone
- Two portfolios with different cash flows can sell for same price because of the nature of perfect substitutes in financial markets

# The Absence of Arbitrage and Financial Market Equilibrium

---

- Implicit Arrow-Debreu prices facilitate transformation of dynamic stochastic choice problems into more easily soluble static problems
  - Time and state contingent commodities
  - Economic role of derivative asset markets
  - Calculation of equilibrium prices in reasonably general models
- Facilitates identification of role of market incompleteness and nontraded assets

# Are These Abstractions Empirically Relevant?

---

- Econometrics is the science of statistical measurement from economic data
- Asset pricing relations should specify what to measure from asset prices and how to measure it
- Perhaps the modern synthesis of Arrow-Debreu contingent claims valuation is the appropriate empirical framework
- Note the Arrow-Debreu foundation of financial engineering

# The Themes of the Course

---

- The tight link between economics and econometrics of asset pricing relations in frictionless arbitrage-free markets
- What can and cannot be learned from asset prices in the transition from general arbitrage-free markets to particular equilibrium models
- The role of smoothness in stochastic processes of economic fundamentals and diversification in asset pricing relations

# Course Outline

---

- Week 1: The Fundamental Theorem of Asset Pricing
- Week 2: State Prices in Alternative Settings
- Week 3: Tautologies and Economics in the Interpretation of the Fundamental Theorem of Asset Pricing
- Week 4: Two Date Asset Pricing Models

# Course Outline Continued

---

- Week 5: The Econometrics of Asset Pricing Relations
- Week 6: Pitfalls in Learning from Asset Prices in Linear Factor Pricing Models
- Week 7: Finite State Approximations to Discrete and Continuous Time Stochastic Processes
- Week 8: Security Price Dynamics

# Course Outline Continued

---

- Week 9: Asymmetric Information, Trade, and Market Efficiency
- Week 10: An Introduction to Market Microstructure from the No-Arbitrage Point of View

# Week One

---

## Asset Markets, Menus, and Payoffs

# Traded and Nontraded Assets

---

- Traded assets include household and business firm liabilities and derivatives
- Thinly- and non-traded assets include many such liabilities and much of real estate and human and physical capital
- Empirical framework must:
  - apply to given asset menu
  - possibly account for relations between assets under study and traded and nontraded assets excluded from analysis

# The Role of Frictionless Markets

---

- Arbitrage-free frictionless markets
  - No taxes, transactions costs, short sales restrictions, or other impediments to trade
  - Need only apply to market participants
  - Any such investor has incentive and ability to exploit perceived arbitrage opportunities in frictionless markets
- All investors in equilibrium models generally act as if markets are frictionless
  - The role of intermediaries in these models

# Notation

- Two dates: today and tomorrow
  - *Nominal* random asset payoffs tomorrow
    - Price + cash flow tomorrow =  $v_i \in \mathcal{V} \subseteq \mathbb{R}^{N(+1)}$
    - Limited liability  $v_s \geq 0$ ,  $s=1, \dots, S$ 
      - $x \geq 0 \Rightarrow x_i \geq 0 \forall i$  and  $x_i > 0$  for some  $i$
- $N (+ 1)$  given assets
  - Initial prices:  $p_i = \psi(v_i)$ ,  $i=1, \dots, N$ 
    - $\psi(\cdot)$  = (linear) pricing operator
  - Nominal returns:  $R_i = v_i / \psi(v_i)$
  - If  $\mathcal{V}$  contains the unit payoff, asset  $N+1$  is riskless asset  $f$  with price  $p_f$  and return  $R_f$

# Some Aspects of the Assumed Market Environment

---

- No presumption that:
  - all assets in menu
  - any investor possesses rational beliefs
  - all investors confront frictionless market
  - time ends tomorrow
- Presumption that some investors:
  - confront no frictions in this market
  - vigorously search for and try to exploit arbitrage opportunities

# Asset Menus

- Primary assets
  - Payoffs potentially range over  $\mathcal{V}$
- Default-free bonds
  - Known nominal payment at each date (i.e.,  $v_{fs}=1$ ;  $s=1, \dots, S$ )
- Risky debt and equity
  - $V_i = B_i + E_i =$  value of firm  $i$  tomorrow
  - $B_i = \min[\mathcal{V}_i, PP_i]$ ;  $PP_i =$  promised payment
  - $E_i = \max[0, \mathcal{V}_i - PP_i]$
- General derivative assets:  $v_d = F[v_i]$

# A Model for Prices in Two Periods of a Given Market

---

- Given  $N (+ 1)$  asset menu
- Two periods, today and tomorrow
- Payoff relevant states
- Perfect markets
  - Frictionless markets
  - Absence of arbitrage
- Agreement on the possible
- Theorem: Strictly positive state prices that price these assets exist iff no arbitrage

# States of Nature in the Arrow-Debreu Model

---

- Many sources of uncertainty impinge on investors and other economic actors
- States of nature in the Arrow-Debreu model provide a complete description of the state of the economy from the perspective of all economic actors
- Accordingly, let  $\Omega$  denote the set of all possible realizations of states of nature
  - In the language of measure theory, the elements of  $\Omega$  comprise the sample space

# *Payoff* Relevant States in *This* Market

---

- States defined entirely by end-of-period assets values in no-arbitrage models
- Seek implications of absence of arbitrage for prices of *this* given menu of *these* assets
- Economic uncertainty characterized by  $\Omega$  impinge on decisions of investors
  - Only relevant for no-arbitrage pricing of *these* assets to extent they describe possible payoffs of *this* given asset menu
  - States defined by their *payoff relevance*

# Assumption One

## Payoff Relevant States

---

- Suppose for now that  $\mathcal{V}$  is a finite set
  - i.e.,  $\mathcal{V} = \{v_{is}; i=1, \dots, N; s=1, \dots, S\}$
- *Assumption 1*: All investors who trade in these assets in this market correctly perceive a deterministic mapping between (finite) end-of-period asset payoffs and underlying states of nature  $s$ 
  - i.e.,  $\{\infty > v_{is} \geq 0, i=1, \dots, N (+1), s=1, \dots, S\}$ 
    - i.e., limited liability need only hold for some  $i$
  - Usual sample space with payoff relevant states

# Discussion of the Frictionless Markets Assumption

---

- No-arbitrage models make few assumptions about *all* investors because *any* investor who prefers more to less:
  - has incentive to vigorously exploit arbitrage opportunities
  - is free to do so in frictionless markets
- Of course, *all* investors can increase wealth by altering their portfolios to exploit such opportunities
  - Unbounded relaxation of budget constraint

# For Whom Must the Market Be Frictionless?

---

- At least one investor unconstrained at the margin with respect to these assets
- Candidates for such investors abound in real world asset markets, including pension funds, endowments, mutual funds, and insurance companies
- If no such marginal investor, absence of arbitrage opportunities places different but related constraints on asset prices

# Assumption 2

## Frictionless Markets

---

- *Assumption 2*: There are no taxes, transactions costs, indivisibilities, short sales restrictions, or other impediments to free trade in the market for these assets for some investors in this market
  - Frictionless markets and the absence of arbitrage opportunities are often called the *Perfect Market Assumptions*

# The Definition of an Arbitrage: The Law of One Price

---

- Weakest such definition is the one that arises the most frequently, the *Law of One Price*
- Two assets with identical payoffs in each state of nature should sell for the same price in the absence of arbitrage
  - $\forall v_1, v_2 \in \mathcal{V}, v_1 = v_2 \Rightarrow \psi(v_1) = \psi(v_2)$

# The Definition of an Arbitrage: No Sure Free Lunches

---

- A *Sure Free Lunch* is zero cost investment that earns positive profits in each state
  - Implicitly involves borrowing at one riskless rate and lending at another via combination of long and short positions in these assets
- No sure free lunch or weak no-arbitrage:
  - $\forall v \in \mathcal{V}, v \geq 0 \Rightarrow \psi(v) \geq 0; v = 0 \Rightarrow \psi(v) = 0$
  - Assets which in all states yield:
    - nonnegative payoffs have weakly positive prices
    - nothing have zero prices

# The Definition of an Arbitrage: No Possible Free Lunches

---

- One potential arbitrage opportunity not excluded by either definition
  - Zero net investment portfolio with no risk of loss and *positive probability* of positive payoff
- No possible free lunch or strong no-arbitrage:
  - $\forall v \in \mathcal{V}, v \geq 0 \Rightarrow \psi(v) > 0$
  - Assets with weakly positive payoffs have strictly positive prices
- $\Rightarrow$  Absence of arbitrage requires no sure or possible free lunches

# The Unconstrained Nature of Beliefs in No Arbitrage Models

---

- No-arbitrage models place only weak restrictions on investor beliefs
  - In particular, no presumption that these investors possess rational expectations
- No-arbitrage models require the inability to make something for nothing, not prices based on rational beliefs or equal to fundamental values in any sense
- Requirement is *agreement on the possible*
  - i.e., investors must agree on *possible states*  $v \in \mathcal{V}$

# The Weak Restrictions on Beliefs in No-Arbitrage Models

---

- Investor who thought some state could not occur would sell infinite amount of claim that pays off only in this state at any positive price to those who thought state could occur
- No equilibrium in market for this claim
  - i.e., no price that would clear this market
- $\Rightarrow$  investors must agree on possible payoffs
  - Agreement on measure zero subsets of  $\mathcal{V}$
  - Beliefs are equivalent probability measures
    - i.e., absolutely continuous wrt one another

# Formal Statement of *Agreement on the Possible*

- Lemma: If no strong arbitrage opportunities:

$$(1) \psi(v^*) = 0 \Rightarrow \Pi(v^*) = 0$$

$$(2) \Pi(v^*) = 0 \Rightarrow \psi(v^*) = 0$$

- Proof

- Strong no arbitrage  $\Rightarrow$  (1) holds by definition

- For (2), assume  $\Pi(v^*) = 0$  and  $\psi(v^*) > 0$ . Buy riskless bond for  $\psi(\mathcal{V})$  and sell  $\psi(\mathcal{V})/\psi(v^*)$  units of asset with payoff  $v^*$ . Cost and payoff are:

$$\text{Cost: } \psi(\mathcal{V}) - [\psi(\mathcal{V})/\psi(v^*)] \psi(v^*) = 0$$

$$\text{Payoff: } 1 - [\psi(\mathcal{V})/\psi(v^*)] \times 0 = 1$$

$\Rightarrow$  arbitrage unless  $\Pi(v^*) > 0$  Copyright © Bruce N. Lehmann 1999-2007

# The Fundamental Theorem of Asset Pricing Theory

- *The Fundamental Theorem of Asset Pricing:* Given assumptions 1-3, a positive pricing rule  $\psi(\bullet)$  supported by a set of positive state prices  $\{\psi_s = \psi(e_s), s=1, \dots, S\}$ , not necessarily unique, exists iff there are no strong arbitrage opportunities (i.e., possible free lunches) in this market
  - $\Rightarrow p_i = \sum_{s \in S} \psi_s v_{is}$
  - $e_s = S$ -vector of zeros except for one in  $s^{\text{th}}$  position
  - $\psi_s =$  price of pure *state contingent claim*
    - pays one unit of account in state  $s$  and zero otherwise

# The Fundamental Theorem: Proof of If

- Economic intuition: budget constraints hold with equality
- Assume  $p_i = \sum_{s \in S} \psi_s v_{is}$ ;  $\{\psi_s > 0, s=1, \dots, S\}$
- Let  $\omega$  be a portfolio with  $v_\omega \geq 0$ . Then:

$$\begin{aligned}\sum_i \omega_i v_{is} &\geq 0 \quad \forall s \implies \sum_i \omega_i v_{is} \psi_s \geq 0 \quad \forall s \\ &\implies \sum_i \omega_i \sum_{s \in S} v_{is} \psi_s \geq 0 \quad \forall s \\ &\implies \sum_i \omega_i \sum_{s \in S} v_{is} \psi_s > 0 \quad \forall s \\ &\implies \sum_i \omega_i p_i > 0\end{aligned}$$

if  $\omega$  does not earn arbitrage profits.

# The Fundamental Theorem: Proof of Only If

- Let  $\mathcal{W} = [-p \ \mathcal{V}] \Rightarrow \mathcal{W}' \omega = 0$  if no arbitrage
  - i.e.,  $\sum_i \omega_i v_{is} = 0 \ \forall s \Rightarrow \sum_i \omega_i p_i = 0$
- Stiemke's lemma: Suppose  $\mathcal{W}' \omega = x$  has no solution  $x \geq 0$  except  $x = 0$ . Then  $\exists \delta \gg 0$  with  $\mathcal{W}\delta = 0$
- $\Rightarrow \exists \delta \gg 0$  s. t.  $\delta_0 (-p) + \sum_{s \in S} v_s \delta_s = 0$ 
  - $v_s$  is the  $N$  vector of asset payoffs in state  $s$
- $\Rightarrow \{ \psi_s = \delta_s / \delta_0 > 0, s=1, \dots, S \}$  s. t.  $p = \mathcal{V}\psi$

# An Aside on Farkas' Lemma and Weak No Arbitrage

---

- Farkas' lemma or duality theorem of linear programming: Suppose  $\mathcal{W}'\omega = x$  has no solution  $x \geq 0$  with  $x_0 > 0$ . Then  $\exists \delta \geq 0$  with  $\mathcal{W}\delta = 0$  and  $\delta_0 = 1$
- $\Rightarrow \{ \psi_{F_s} = \delta_s \geq 0, s=1, \dots, S \}$  s. t.  $p = \mathcal{V}\psi_F$
- Strict positivity of state prices requires the absence of both sure or possible free lunches

# Uncountably Infinite State Spaces

---

- Partition  $\mathcal{V}$  into  $S(\varepsilon) < \infty$ ,  $S(\varepsilon)$  decreasing in  $\varepsilon$ , disjoint regions  $\{\mathcal{V}_s, s=1, \dots, S(\varepsilon)\}$ 
  - i.e.,  $\mathcal{V} = \cup_{S(\varepsilon)} \mathcal{V}_s$
- No-arbitrage  $\Rightarrow 0 < q(\mathcal{V}_s) < 1 \quad \forall \varepsilon > 0$ 
  - i.e.,  $0 < \psi(\mathcal{V}_s) < \infty \quad \forall \varepsilon > 0$
- $\Rightarrow \lim_{\varepsilon \rightarrow 0} Q_{S(\varepsilon)} = Q(\bullet)$ 
  - “...every sequence that tries to converge is successful” in a complete metric space
- Limiting  $Q$  and  $\psi$  are smooth if no arbitrage

# The Approximation of a Continuous Random Variable

- Let  $x \sim F(x)$  be a continuous random variable
- Consider the discrete approximation:
  - $$x_n = \sum_{k \in \{1, n2^n\}} [(k-1)2^{-n}] 1_{(k-1)2^{-n} \leq x < k2^{-n}} + n 1_{x > n} \\ - \sum_{k \in \{2, n2^n\}} [(k-1)2^{-n}] 1_{-k2^{-n} < x \leq -(k-1)2^{-n}} - n 1_{x < -n}$$
  - By construction:
    - (i)  $x = \lim_{n \rightarrow \infty} x_n \quad \forall x$
    - (ii) If  $x \geq 0 \Rightarrow x \geq x_{n+1} \geq x_n \Rightarrow x_n \uparrow x$ 
      - $\Rightarrow |x_n - x| = x - x_n \downarrow 0$
    - (iii) If  $x < 0 \Rightarrow x \leq x_{n+1} \leq x_n \Rightarrow x_n \downarrow x$ 
      - $\Rightarrow |x_n - x| = x_n - x \downarrow 0$
    - (iv)  $|x| \leq m \Rightarrow |x_n - x| \leq 2^{-n} \quad \forall n > m$

# The Fundamental Theorem of Digital Television

- The multinomial approximation
  - $F_n(n) = F(1_{x>n})$
  - $F_n[(k-1)2^{-n}] = F(1_{(k-1)2^{-n} \leq x < k2^{-n}})$
  - $F_n[-(k-1)2^{-n}] = F(1_{-k2^{-n} < x \leq -(k-1)2^{-n}})$
  - $F_n(-n) = F(1_{x < -n})$
- Suppose  $F$  is tight
  - $\forall \varepsilon > 0 \exists n(\varepsilon) < \infty$  s. t.  $F(1_{x>n(\varepsilon)}) + F(1_{x<n(\varepsilon)}) < \varepsilon$
- $\Rightarrow F_n(x) \downarrow F(x)$  by construction
  - $F_n(x) \uparrow F(x)$  if  $k$  replaces  $k-1$  in  $F_n$

# A Measure-Theoretic Version of the Multinomial Approximation

---

- Part 1: The Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$  of the real line  $\mathbb{R}$  (i.e.,  $-\infty < x < \infty$ ) is generated by the half-open intervals:
  - $(k-1)2^{-n} \leq x < k2^{-n}$ 
    - Open, closed, finite, and semi-infinite intervals, too
- Part 2: The rational numbers are countable and so:
  - they are dense in the line, making  $\mathbb{R}$  separable
  - provide a countable set of rational endpoints (i.e., rational  $k$  and  $n$ ), yielding a countable base

# Measure-Theoretic Reformulation, Continued

- Part 3:  $\Rightarrow$  All  $\Pi \in \mathcal{P}$  are tight where  $\mathcal{P}$  is the set of probability measures on  $\mathcal{B}(\mathbb{R})$ 
  - $\mathbb{R} = \cup_s [a_s, b_s[ \Rightarrow 1 = \Pi(\mathbb{R}) = \cup_s \Pi([a_s, b_s[)$
  - $\Rightarrow \forall \varepsilon > 0 \exists \{b_0, a_s\}$  s. t.  $\Pi([-\infty, b_0[) + \Pi([a_s, -\infty[) < \varepsilon$
  - i.e., tightness requires only that the mass of  $\Pi$  not disappear to either infinity
- Proof for  $\mathbb{R}$  goes through for  $\mathbb{R}^{N(+1)}$  by replacing intervals with hyperrectangles

# Multinomial Approximation on More General Spaces

- Similar proof for complete separable metric space: Ulam's Theorem (Dudley, p. 176)
  - $\mathcal{V}$  can be covered by open  $1/n$  balls for each  $n$
  - Separability  $\Rightarrow \mathcal{V}$  can be covered by a countable set of closed (or open)  $1/n$  balls  $\{\mathcal{V}_{ns}, s=1,2,\dots\}$
  - $\Rightarrow \forall \varepsilon > 0 \exists S_n$  s. t.  $\Pi(\mathcal{V} \setminus \mathcal{V}_n) < \varepsilon 2^{-n}$ 
    - $\mathcal{V}_n = \bigcup_{s \leq S_n} \mathcal{V}_{ns} \subseteq \mathcal{V}; \mathcal{V}_{n+1} \subseteq \mathcal{V}_n \forall n$  by construction
    - $\mathcal{V}$  complete  $\Rightarrow$  totally bounded set  $\bigcap_n \mathcal{V}_n$  has compact closure
  - Definition: Finite measure  $\Pi$  on  $\mathcal{B}(\mathcal{V})$  is tight if  $\forall \varepsilon > 0 \exists$  a compact set  $K_\varepsilon$  s. t.  $\Pi(K_\varepsilon) > 1 - \varepsilon$

# Multinomial Approximation on More General Spaces, Continued

- Ulam's theorem, continued
  - $\Rightarrow \Pi(\mathcal{V} \cap_n \mathcal{V}_n) \leq \sum_n \Pi(\mathcal{V} \setminus \mathcal{V}_n) < \sum_n \varepsilon 2^{-n} < \varepsilon$
  - $\Rightarrow \Pi$  is tight
- General multinomial construction
  - Carve compact set  $\mathcal{V}_n$  into  $S(\varepsilon)-1 < \infty$  disjoint cells  $\mathcal{V}_s$ ,  $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset \forall i \neq j$ , and put  $\mathcal{V}_S = \mathcal{V} \setminus \mathcal{V}_n$
  - $\Rightarrow$  tight sequence  $\Pi_S(A)$  converges uniformly to  $\Pi(A)$  by construction  $\forall A \in \mathcal{V}$  and  $\Pi \in \mathcal{P}$
  - i.e.,  $\Pi_S(A) - \Pi(A) = \sum_s \Pi(\mathcal{V}_s) 1_{A \cap \mathcal{V}_s} - \Pi(A)$   
 $= \sum_s [\Pi(\mathcal{V}_s) - \Pi(A \cap \mathcal{V}_s)] 1_{A \cap \mathcal{V}_s} \downarrow 0$

# Approximation via the Henstock-Kurzweil Gauge Integral

---

- Gauge integrals replace fixed width sums of Riemann integral with variable width sums
  - This slight redefinition results in an integral that is *more* general than a Lebesgue integral
  - Uses Riemann idea of approximating the domain of the function being approximated
  - Lebesgue integral approximates its range
- Virtue: no need for measure theory!
- Defect: Fubini and Tonelli theorems harder
  - Probably only a bit harder for  $\mathbb{R}^{N(+1)}$

# Approximation via the Henstock-Kurzweil Gauge Integral, Cont'd

- Definitions

- Tagged partition of  $[a,b]$ : set of numbers  $x_0, \dots, x_n$ ,  $z_1, \dots, z_n$  with  $a = x_0 \leq z_1 \leq x_1 \leq \dots \leq z_n \leq x_n = b$

- $\delta$  partition:  $m(P) < \delta$  if  $x_k - x_{k-1} < \delta$  for all  $k$

- Riemann sum for  $f$  with respect to  $P$ :

$$S(f, P) = \sum_k f(z_k)(x_k - x_{k-1})$$

- Riemann integrability of  $f$ : if  $\exists A$  s.t.  $\forall \varepsilon > 0 \exists \delta > 0$   
 $\forall P$   $m(P) < \delta$  implies  $|S(f,P) - A| < \varepsilon$

- Gauge on  $[a,b]$ : function  $\delta: [a,b] \rightarrow (0, \infty)$

- $\delta$ -fine partition:  $P$  s.t.  $m(P) < \delta$  if  $x_k - x_{k-1} < \delta(z_k) \forall k$

- i.e.,  $x_k - x_{k-1} < \delta(z_k)$  replaces  $x_k - x_{k-1} < \delta$

# Approximation via the Henstock-Kurzweil Gauge Integral, Cont'd

- Definition
  - Gauge integrability of  $f$ : if  $\exists A$  s.t.  $\forall \varepsilon > 0 \exists$  gauge  $\delta$  s.t.  $\forall \delta$ -fine tagged partitions  $P$   $m(P) < \delta$  implies  $|S(f,P) - A| < \varepsilon$
  - Clearly gauge  $\Rightarrow$  Riemann integrability
- Cousin's Lemma:  $\forall \delta \exists P$  with  $m(P) < \delta$ .
- Proof. Heine-Borel open cover theorem compactness argument using LUB property
  - Take  $Y = \{y: \exists P \text{ a partition of } [a,y], m(P) < \delta\}$ .  
 $\Rightarrow \sup(Y) \in Y$  and  $\sup S < b \Rightarrow$  a contradiction.

# Approximation via the Henstock-Kurzweil Gauge Integral, Cont'd

---

- Gauge integration theorem:  $f$  is gauge-integrable if  $f$  is continuous.
- Proof: Cousin's lemma guarantees existence of  $\delta$ -fine partition and then do variable width Riemannian sums with  $[x_k, x_{k-1}] = z_k \pm \delta(z_k)$ .
  - Notice no requirement of uniform continuity
- Theorem. If  $f$  is differentiable on  $[a, b]$ ,  $f'$  is gauge-integrable and  $\int f' = f(b) - f(a)$ .
  - Note no requirement that  $f'$  is integrable!

# Approximation via the Henstock-Kurzweil Gauge Integral, Cont'd

- Proof: Examine:
  - $S(f',P) - [f(b) - f(a)] = \sum_k f'(z_k)(x_k - x_{k-1}) + f(z_k) - (f(z_{k-1}))$
  - Works if  $f$  not differentiable at countable number of points
- $f$  is Lebesgue integrable iff  $f$  and  $|f|$  are gauge integrable
- I do not completely understand the extension to  $\mathbb{R}^{N(+1)}$
- My point: the fundamental theorem of digital TV works in general without measure theory

# Sufficient Conditions for Tightness

- Uniform tightness
  - Sequence  $\{\Pi_n, n=1, \dots\}$  on a metric space is uniformly tight if  $\forall \varepsilon > 0 \exists$  a compact set  $K_\varepsilon$  s. t.  $\lim_{n \rightarrow \infty} \sup \Pi_n[|x_n| \notin G] < \varepsilon \forall$  open  $G \supseteq K_\varepsilon$
  - $x_n \in \mathbb{R}^N = O_p(1) \Rightarrow \forall \varepsilon > 0 \exists K_\varepsilon, n(\varepsilon) < \infty$  s.t.  $\Pi[|x_{n(\varepsilon)}| > K_\varepsilon] < \varepsilon \Rightarrow \lim_{n \rightarrow \infty} \sup \Pi[|x_n| \notin G] \leq \varepsilon$ 
    - $G$  is any open set in closed ball  $\{x: |x| \leq K_\varepsilon\}$
- Uniform integrability:  $\{x_t, t \in T\}$  is uniformly integrable if  $\lim_{c \rightarrow \infty} \sup E[|x_n| 1_{|x_t| > c}] = 0$ 
  - Sufficient if  $\exists r > 0$  s.t.  $\lim_{n \rightarrow \infty} \sup E[|x_n|^r] < \infty$

# Implications for State Pricing on Uncountable State Spaces

- Continuous mapping theorem insures continuity of state pricing operators
  - $f: \mathbb{R}^{N(+1)} \rightarrow \mathbb{R}$  is continuous iff  $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f$
- *The Fundamental Theorem of Asset Pricing* holds on infinite dimensional state spaces
  - Risk neutral measures  $Q$  lie in  $\mathcal{P}$ !
    - $Q(A) = \int_A \psi(v) dv / \int_V \psi(v) dv$  is finite Borel measure in  $\mathcal{P}$
    - i.e., Radon-Nikodym theorem for risk neutral measures
    - i.e., limiting sequence of transformations from multinomial probability approximations to finite dimensional state prices
- Price systems linear in frictionless markets

# Some of the Economic Issues in the Fundamental Theorem

---

- Incomplete markets and the set of state prices
- Options and spanning
- Binomial option pricing
- Modigliani-Miller and value additivity
- Numeraires
- A digression on asymmetric information

# Spanning and Market Completeness

- Definition of spanning and completeness
  - Any pattern of payoffs across these states in this market can be achieved with a portfolio of (or trading strategy in) these assets
    - $\forall \mathbf{x} = \{x_1, \dots, x_S\} \exists \omega = \{\omega_1, \dots, \omega_N\}$  s.t.  $\sum_i \omega_i p_i < \infty$ ;  
 $\sum_s \omega_i v_{is} = x_s$
    - $\text{Rank}(\mathcal{V}) = S$
    - State prices are unique:  $\psi = [\mathcal{V}'\mathcal{V}]^{-1}\mathcal{V}'p$
- Warning: not a statement about Pareto efficiency which requires  $\text{Rank}(\Omega) = S$  and exhaustive asset menu

# The Economics of Spanning for Payoff Relevant States

---

- Suppose:
  - *payoff relevant* state  $s$  not spanned
    - i.e., cannot insure against occurrence of  $s$ 
      - $\nexists \omega$  s. t.  $\mathcal{V}'\omega = e_s$
  - at least two investors have different shadow prices for claim  $e_s$
- $\Rightarrow$  Financial engineer earns arbitrage profit by selling it to high valuation investor and buying claim  $e_s$  from low valuation investor
  - Competition among engineers should eliminate rents

# The Economics of Spanning for Payoff Relevant States, Continued

---

- No barriers to free supply of butterflies
  - No lack of observability or moral hazard to complicate the writing of derivative contracts
  - No frictions makes it feasible
  - Profit motive provides incentive
- $\Rightarrow$  One should probably assume that markets are complete in a payoff relevant sense

# Options

---

- Call options
  - Right to buy asset  $i$  at prespecified price  $E$
  - Payoff =  $\max[0, v_i - E]$
- Put options
  - Right to sell asset  $i$  at prespecified price  $E$
  - Payoff =  $\max[0, E - v_i]$
- Asset  $i$  is called the underlying asset
- $E$  is called the exercise or strike price
- N. B.: these are single period options

# Options and Spanning

- Algorithm for insuring state  $s$ 
  - Either:
    - Reorder the columns of  $\mathcal{V}$  s. t.  $\exists i v_{is-1} < v_{is} < v_{is+1}$
    - Form portfolio  $\omega$  s. t.  $v_{\omega s-1} < v_{\omega s} < v_{\omega s+1}$
    - Denote these three payoffs as  $v_{s-1} < v_s < v_{s+1}$
    - Note collinearity of  $v_s$  with remainder of  $\mathcal{V}$  not likely *a priori* due to payoff relevance of states
  - Terminal call option values at strike price:
    - $E = v_{s-1}$ :  $\max[0, v - v_{s-1}]$
    - $E = v_s$ :  $\max[0, v - v_s]$
    - $E = v_{s+1}$ :  $\max[0, v - v_{s+1}]$

# Options and Spanning, Continued

---

- Consider two option portfolios
- Portfolio A: long strike  $s-1$  and short strike  $s$ 
  - Payoff =  $\max[0, v - v_{s-1}] - \max[0, v - v_s]$ 
    - = 0 if  $v \leq v_{s-1}$
    - =  $v_s - v_{s-1}$  if  $v > v_{s-1}$
- Portfolio B: long strike  $s+1$  and short strike  $s$ 
  - Payoff =  $\max[0, v - v_{s+1}] - \max[0, v - v_s]$ 
    - = 0 if  $v \leq v_s$
    - =  $v_s - v_{s+1}$  if  $v > v_{s+1}$

# Options and Spanning, Continued

---

- Butterfly spread (weighted):
  - long  $1/[v_s - v_{s-1}]$  of Portfolio A
  - long  $1/[v_{s+1} - v_s]$  of Portfolio B
- Butterfly payoff
  - 0 if  $v \leq v_{s-1}$
  - $[1/(v_s - v_{s-1})] \times (v_s - v_{s-1}) = 1$  if  $v = v_s$
  - $1 + [1/(v_{s+1} - v_s)] \times (v_s - v_{s+1}) = 0$  if  $v > v_s$
- Butterfly spread = pure state contingent claim

# Butterfly Prices and the State Price Density

- Butterfly price
  - $p_b = [C(p, v_s - \Delta v) - 2C(p, v_s) + C(p, v_s + \Delta v)] / \Delta v$   
 $= \{ [C(p, v_s - \Delta v) - 2C(p, v_s) + C(p, v_s + \Delta v)] / (\Delta v)^2 \} \Delta v$
- Limiting butterfly prices
  - $\Rightarrow \lim_{\Delta v \rightarrow 0} p_b(v_s) = \{ \partial^2 C(p, E) / \partial E^2 \} |_{E=v_s} dv$  is the state price density
  - $\Rightarrow$  State price density is C2
- Limiting price of arbitrary derivatives
  - $p_{f(v)} = \int f(v) p_b(v) = \int f(v) [\partial^2 C(p, v) / \partial v^2] dv$

# The Linear Pricing Rule and the Modigliani-Miller Theorem

- Let  $L_{ijs}$  be the value of liability  $j$  of firm  $i$  tomorrow
  - $V_{\text{firm}} = \sum_s \sum_j \psi_s L_{ijs} = \sum_s \psi(L_{ijs}) = \psi[\sum_s L_{ijs}]$
  - Firm value independent of liability structure
  - Important result known as *value additivity*
- Important caveat: span of market — that is, linear pricing operator  $\psi(\bullet)$  — is unchanged by issuance of liabilities by this firm

# Can the Firm Earn Monopoly Rents by Selling Risky Debt?

---

- In principle, yes if:
  - the firm has the monopoly right to issue contingent claims that span these return states
  - households demand such claims and do not have perfect substitutes for them
- In practice, no because:
  - Zero net supply derivative contracts like options are easy to create and enforce, breaking potential monopoly power
  - Likely many close substitutes on demand side

# Alternative Numeraires: The Nominal/Real Distinction

---

- Traded assets typically denominated in units of account
- Absence of arbitrage a property of price system irrespective of numeraire
  - could base real returns on:
    - some price deflator (popular empirically)
    - $p_f$  (popular in general equilibrium finance)
- Absence of arbitrage restricts nominal asset prices because investors prefer more nominal wealth to less

# Alternative Numeraires: International Arbitrage Pricing

---

- Changes of numeraire arise naturally in an international setting
- Positive state prices must exist in every currency of denomination so long as foreign exchange markets satisfy the *Perfect Market Assumptions*
  - $\{\psi_{\text{foreign},s}, s=1,\dots,S\} = \{e_s \psi_{\text{domestic},s}, s=1,\dots,S\}$ 
    - $e_s$  = foreign/domestic currency
    - State space must be augmented to accommodate exchange rate states if necessary

# Heterogeneous Information and State Prices

---

- Reasonable suppositions
  - Some investors better informed than others
  - Many (most?) investors better informed than econometric students of asset markets
- Implications
  - Refine payoff relevant states into states with differing information substates
    - termed finer state partitions
  - (Some) investors might have more accurate probability estimates than others

# Asymmetric Information and No-Arbitrage Pricing

- Prices, not probabilities, are the issue
- Investors with superior information still must perceive no arbitrage opportunities
  - An investor  $j$  who can:
    - trade without frictions in these assets
    - partition state  $s$  into disjoint  $K$  *possible* substates
  - must perceive no arbitrage possibilities so:
    - $\psi_s = \sum_k \psi_{jsk}$ 
      - $\psi_{jsk}$  = investor  $j$ 's perceived state price for substate  $k$
      - $\psi_s$  = arbitrage-free payoff relevant state price
        - Only one such price if this market is spanned